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1967

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Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1967. Vol. 174, No. 5

UDC 517.933

MATHEMATICS

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OPTIMALITY CONDITIONS IN CONTROL PROBLEMS WITH INTERMEDIATE CONDITIONS

(Presented by Academician L. S. Pontryagin, 5 VIII 1966)

The results of L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, generalized by the authors in the monograph ⁽¹⁾, make it possible to formulate optimality conditions for a broad class of problems in the theory of optimal processes. In the present paper, which to a considerable extent uses the methods set forth in ⁽¹⁾, and some results of the work ⁽²⁾, optimality conditions are obtained for the following problem.

1°. Statement of the problem. Consider the n -dimensional system of equations

$$\dot{x}(t) = f(x, u, t), \quad (1)$$

whose trajectory must satisfy the endpoint conditions

$$A_\alpha[x(T_0), T_0] = 0, \quad \alpha = 0, 1, \dots, a \leq n, \quad (2)$$

$$B_\beta[x(T), T] = 0, \quad \beta = 0, 1, \dots, b < n, \quad (3)$$

and, for $t = T_1$, $T_0 < T_1 < T$, the intermediate conditions

$$C_\gamma[x(T_1), T_1] = 0, \quad \gamma = 0, 1, \dots, c < n. \quad (4)$$

In what follows, assuming that

$$dA_0(x, t)/dt|_{t=T_0} \neq 0, \quad dB_0(x, t)/dt|_{t=T} \neq 0, \quad dC_0(x, t)/dt|_{t=T_1-0} \neq 0,$$

we shall use the conditions

$$A_0[x(T_0), T_0] = 0, \quad B_0[x(T), T] = 0, \quad C_0[x(T_1), T_1] = 0 \quad (5)$$

to determine the instants of time T_0 , T , and T_1 .

As the class of admissible controls we take r -dimensional piecewise-continuous vector functions $u(t)$, whose values must belong to a prescribed closed domain U .

Let the functional

$$I(x, u) = \Phi[x(T), T]. \quad (6)$$

We pose the following problem. **Among all controls $u(t) \in U$ such that the trajectories of system (1) corresponding to these controls satisfy conditions (2), (3), and (4), choose one for which the functional (6) takes the minimal value.**

The functions $A_\alpha(x, t)$, $B_\beta(x, t)$, $C_\gamma(x, t)$, and $\Phi(x, t)$ are assumed to be continuous together with their first-order partial derivatives and to have bounded second-order partial derivatives with respect to the arguments x, t . The vector function $f(x, u, t)$ will be assumed to have the indicated properties with respect to the aggregate of the arguments x, u and to be continuous in t .

2°. Optimality conditions. Introduce the notation

$$A(x, t) = \sum_{\alpha=1}^a \lambda_\alpha^A A_\alpha(x, t), \quad B(x, t) = \sum_{\beta=1}^b \lambda_\beta^B B_\beta(x, t) + \lambda^\Phi \Phi(x, t),$$

$$C(x, t) = \sum_{\gamma=1}^c \lambda_\gamma^C C_\gamma(x, t).$$

Theorem 1 (maximum principle). If the control $u(t)$ and the trajectory $x(t)$ are optimal in problem (1)–(6), then there exist numbers λ_α^A , λ_β^B , λ_γ^C , and λ^Φ satisfying the conditions

$$\sum_{\alpha=1}^a (\lambda_\alpha^A)^2 + \sum_{\beta=1}^b (\lambda_\beta^B)^2 + \sum_{\gamma=1}^c (\lambda_\gamma^C)^2 + (\lambda^\Phi)^2 = 1, \quad \lambda^\Phi \geq 0,$$

and such a vector-function $p(t)$, satisfying the system of equations ⁽¹⁾

$$\dot{p}(t) = -\text{grad}_x H(x, p, u, t), \quad T_0 \leq t < T_1, \quad T_1 < t \leq T, \quad (7)$$

where $H(x, p, u, t) \equiv (p, f(x, u, t))$, the boundary conditions

$$p(T_0) = \left[\text{grad}_x A(x, t) - \left(\frac{dA(x, t)}{dt} \Big/ \frac{dA_0(x, t)}{dt} \right) \text{grad}_x A_0(x, t) \right]_{t=T_0}, \quad (8)$$

$$p(T) = \left[-\text{grad}_x B(x, t) + \left(\frac{dB(x, t)}{dt} \Big/ \frac{dB_0(x, t)}{dt} \right) \text{grad}_x B_0(x, t) \right]_{t=T} \quad (9)$$

and the jump condition

$$p(T_1 - 0) - p(T_1 + 0) \quad (10)$$

$$= \left[-\text{grad}_x C(x, t) + \left(\left[\frac{dC(x, t)}{dt} - \mu \right] \Big/ \frac{dC_0(x, t)}{dt} \right) \text{grad}_x C_0(x, t) \right]_{t=T_1-0},$$

where

$$\mu = (p(T_1 + 0), [f(x(T_1), u(T_1 - 0), T_1) - f(x(T_1), u(T_1 + 0), T_1)]),$$

such that the maximum condition is fulfilled

$$H(x, p, u, t) = \sup_{v \in U} H(x, p, v, t), \quad T_0 < t < T_1, \quad T_1 < t < T. \quad (11)$$

The method of proving Theorem 1 and the assertions formulated below is based on studying cones of attainability ⁽¹⁾ in an $(a + b + c + 1)$ -dimensional vector space, along whose axes are laid off variations of the functional (6) and of the left-hand sides of the equalities (3), (4), and (2) for $\beta = 1, \dots, b$, $\gamma = 1, \dots, c$, and $\alpha = 1, \dots, a$, calculated under the condition that, for the varied trajectory ⁽¹⁾ of system (1), the second, third, and first equalities (5), respectively, are satisfied. The indicated variations can be written in a form analogous to that used in ⁽²⁾, with a remainder term whose estimate, in the case of a fixed left endpoint of the trajectory, coincides with that obtained in ⁽²⁾.

Theorem 2. For optimality of the control $u(t)$ “in the small” ⁽²⁾ on an interval $[\tau_1, \tau_2]$, containing none of the time instants T_1 and T , in problem (1)–(6) with fixed left endpoint of the trajectory, it is sufficient that there exist numbers λ_β^B , λ_γ^C , and λ^Φ such that $\lambda^\Phi > 0$ and the function $H(x, p, u, t)$, determined by conditions (7), (9), and (10), satisfies the maximum condition (11) on the interval $[T_0, T]$ and the conditions of Theorem 1 of the work ⁽²⁾ on the interval $[\tau_1, \tau_2]$.

3°. Linear systems. Let system (1) be linear in x ,

$$\dot{x}(t) = F(t)x + \varphi(t, u), \quad (12)$$

the time instants T_0 , T_1 , and T be fixed and the conditions (2)–(4) and the function-

were given in (6) in the form

$$\begin{aligned} A_\alpha[x(T_0), T_0] &\equiv (l_\alpha^A, x(T_0)) + m_\alpha^A = 0, \quad \alpha = 1, \dots, a, \\ B_\beta[x(T), T] &\equiv (l_\beta^B, x(T)) + m_\beta^B = 0, \quad \beta = 1, \dots, b, \\ C_\gamma[x(T_1), T_1] &\equiv (l_\gamma^C, x(T_1)) + m_\gamma^C = 0, \quad \gamma = 1, \dots, c, \\ \Phi[x(T), T] &\equiv (l^\Phi, x(T)), \end{aligned} \quad (13)$$

where the vectors l and the numbers m are constants.

Theorem 3. *For the optimality of the control $u(t)$ and the trajectory $x(t)$ in problem (12), (13), it is sufficient that there exist numbers λ_α^A , λ_β^B , λ_γ^C , and λ^Φ , $\lambda^\Phi > 0$, and such a vector-function $p(t)$, satisfying conditions (7)–(10), that the maximum condition (11) is fulfilled.*

Let $X(t)$ be a fundamental matrix of the system of solutions of the homogeneous system of equations corresponding to system (12). We shall call problem (12), (13), for which $a + b + c = n$ and the vectors $X'(T_0)l_\alpha^A$, $\alpha = 1, \dots, a$, $X'(T_1)l_\gamma^C$, $\gamma = 1, \dots, c$, and $X'(T)l_\beta^B$, $\beta = 1, \dots, b$, are linearly independent, **simplest**.

Theorem 4. *For simplest problems, the conditions of Theorem 3 are sufficient and necessary.*

The results obtained are easily generalized to the case where there are several systems of intermediate conditions of the form (4), and also to the case where $T_1 > T$.

4°. Equations with discontinuous right-hand sides. The following problem (1) reduces to the formulation considered. Suppose that system (1) has a discontinuity at the time $t = T_1$, determined by the fulfillment of the condition $C_0[x(T_1), T_1] = 0$, so that

$$\dot{x}(t) = f^-(x, u, t) \quad \text{for } t < T_1, \quad \dot{x}(t) = f^+(x, u, t) \quad \text{for } t > T_1. \quad (14)$$

Assuming that the left and right systems (14) determine the variation of different phase coordinates $x^-(t)$ and $x^+(t)$, and requiring fulfillment of the conditions

$$C_\gamma[x^-(T_1), x^+(T_1)] \equiv x_\gamma^-(T_1) - x_\gamma^+(T_1) = 0, \quad \gamma = 1, \dots, n,$$

we arrive at a problem with intermediate conditions.

Theorem 5. *If the control $u(t)$ and the trajectory $x(t)$ of system (14) are optimal in the problem of minimizing the functional (6) under conditions (2) and (3), then there exist numbers λ_α^A , λ_β^B , and λ^Φ , satisfying the condition*

$$\sum_{\alpha=1}^a (\lambda_\alpha^A)^2 + \sum_{\beta=1}^b (\lambda_\beta^B)^2 + (\lambda^\Phi)^2 = 1, \quad \lambda^\Phi \geq 0,$$

and such a vector-function $p(t)$, satisfying conditions (7)–(9) and the jump condition

$$p(T_1 - 0) - p(T_1 + 0) = -\nu \operatorname{grad}_x C_0[x(T_1), T_1],$$

where

$$\nu = \frac{(p(T_1 + 0), [f^-[x(T_1), u(T_1 - 0), T_1] - f^+[x(T_1), u(T_1 + 0), T_1]])}{(\operatorname{grad}_x C_0[x(T_1), T_1], f^-[x(T_1), u(T_1 - 0), T_1]) + \partial C_0[x(T_1), T_1]/\partial T_1},$$

that the maximum condition (11) is fulfilled.

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Received

26 VII 1966

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