

ON FUNDAMENTAL SOLUTIONS OF SINGULAR PARTIAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

V. I. Kononenko

1967

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196701.71824>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.944

MATHEMATICS

V. I. KONONENKO

ON FUNDAMENTAL SOLUTIONS OF SINGULAR PARTIAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

(Presented by Academician I. N. Vekua on 19 III 1966)

Let D be a domain of $(n+1)$ -dimensional Euclidean space adjoining the hyperplane $x_{n+1} = 0$. In this domain consider a linear differential operator of order $2m$ of B -elliptic type ⁽¹⁾

$$\mathcal{L}(D_x, B_{x_{n+1}}) = \sum_{\nu=0}^m \sum_{i=0}^{2m-2\nu} \sum_{j_1, \dots, j_i=1}^n A_{j_1, \dots, j_i}^{\nu}(x) \frac{\partial^i}{\partial x_{j_1} \dots \partial x_{j_i}} B_{x_{n+1}}^{\nu}. \quad (1)$$

Here, as usual, $B_{x_{n+1}}$ denotes the Bessel differential operator

$$B_{x_{n+1}} = \frac{\partial^2}{\partial x_{n+1}^2} + \frac{k}{x_{n+1}} \frac{\partial}{\partial x_{n+1}},$$

where $k > 0$, $x_{n+1} \geq 0$.

In paper ⁽²⁾ a fundamental solution of the homogeneous operator (1) was constructed in the case of constant coefficients. In the present note we construct a fundamental solution for the nonhomogeneous operator (1) with constant coefficients. In addition, we prove the existence in the small of a fundamental solution of this operator with variable coefficients. The case $m = 1$ was studied earlier by A. Weinstein ⁽⁶⁾ and others.

1. Let $f(x) = f(x_1, \dots, x_n, x_{n+1})$ belong to the class C_2 and be finite in the half-space R_{n+1}^* ($x_{n+1} \geq 0$). Then this function admits the following expansion in weighted plane waves:

$$f(x) = c_1(u, k, \nu) \Delta_B^{(\nu+\gamma)/2} \int_{R_{n+1}^*} f(y) \left(\int_{\Omega_+} T_x^y |x \cdot \omega|^\nu \omega_{n+1}^k d\omega \right) y_{n+1}^k dy, \quad (2)$$

where Ω_+ is the hemisphere $\sum_1^{n+1} \omega_i^2 = 1$, $\omega_{n+1} \geq 0$; T_x^y is the generalized translation operator ⁽³⁾

$$T_x^y f(x) = c(k) \int_0^\pi f(y_1 - x_1, \dots, y_n - x_n, \sqrt{x_{n+1}^2 + y_{n+1}^2 - 2x_{n+1}y_{n+1} \cos \alpha}) \sin^{k-1} \alpha d\alpha;$$

Δ_B is the Beltrami operator

$$\Delta_B = \sum_1^n \frac{\partial^2}{\partial x_i^2} + B_{x_{n+1}}.$$

An essential role in formula (2) is played by the integral of the form

$$|x \cdot \omega|_B^\nu = \int_0^\pi \left| \sum_1^n x_i \omega_i + x_{n+1} \omega_{n+1} \cos \alpha \right|^\nu \sin^{k-1} \alpha d\alpha,$$

where $\gamma = n+1+k$ and ν is the addition to the number γ up to an even number.

In the case of integer k , expansion (2) has the form

$$f(x) = c_2(n, k, \nu) \Delta_B^{(\nu+\gamma)/2} \int_{R_{n-1}^+} f(y) \left(\int_{\Omega_+} T_x^y |x \cdot \omega|_B^\nu \ln |x \cdot \omega|_B \omega_{n+1}^k d\omega \right) y_{n+1}^k dy. \quad (3)$$

To obtain formulas (2) and (3), the method of F. John is used [4].

2. Let $\mathcal{L}(D_x, B)$ be a B -elliptic operator with constant coefficients. The problem of finding the fundamental solution of the operator $\mathcal{L}(D_x, B_{x_{n+1}})$ is equivalent to solving the inhomogeneous equation

$$\mathcal{L}(D_x, B_{x_{n+1}})u = f(x) \quad (4)$$

for an arbitrary right-hand side $f(x)$. In view of the validity of expansions (2) and (3), it is sufficient to solve the equation

$$\mathcal{L}(D_x, B_{x_{n+1}})u = |x \cdot \omega|_B^\nu.$$

We seek the solution of this equation in the form

$$u = \int_0^\pi v \left(\sum_1^n x_i \omega_i + x_{n+1} \omega_{n+1} \cos \alpha \right) \sin^{k-1} \alpha d\alpha.$$

As a result, for the function $v(\xi)$ we obtain an ordinary differential equation, whose solution is written with the aid of the Duhamel integral. Then the fundamental solution of equation (4) will have the form

$$K(x) = c \Delta_B^{(\nu+\gamma)/2} \int_{\Omega_+} \int_0^\pi \int_0^\xi \int_C \frac{e^{(\xi-\tau)\lambda}}{\mathcal{L}(\omega\lambda)} d\lambda g(\tau) d\tau \sin^{k-1} \alpha d\alpha \omega_{n+1}^k d\omega, \quad (5)$$

where

$$\xi = \sum_1^n x_i \omega_i + x_{n+1} \omega_{n+1} \cos \alpha,$$

$$g(\tau) = \begin{cases} |\tau|^\nu \ln |\tau|, & \text{if } \gamma \text{ is even,} \\ |\tau|^\nu, & \text{in all other cases.} \end{cases}$$

Here C is a contour in the complex λ -plane enclosing all roots of the equation $\mathcal{L}(\omega\lambda) = 0$. The existence of such a standard contour for all $\omega = (\omega_1, \dots, \omega_{n+1})$ is ensured by the B -ellipticity of the operator $\mathcal{L}(D_x, B_{x_{n+1}})$. To obtain a fundamental solution with a singularity at an arbitrary point z , one must apply the shift operator T_x^z to the function $K(x)$. In the case of the homogeneous operator $\mathcal{L}(D_x, B_{x_{n+1}})$, the contour integral in formula (5) is readily computed. As a result, the fundamental solution has the form

$$K(x) = c \Delta_B^{(\nu+\gamma)/2} \int_{\Omega_+} \int_0^\pi |\xi|^{2m+\nu} \sin^{k-1} \alpha d\alpha \frac{\omega_{n+1}^k}{\mathcal{L}(\omega)} d\omega,$$

if γ is not an even number. If, however, γ is an even number, then

$$K(x) = c \Delta_B^{\gamma/2} \int_{\Omega_+} \int_0^\pi \xi^{2m} \ln |\xi| \sin^{k-1} \alpha d\alpha \frac{\omega_{n+1}^k}{\mathcal{L}(\omega)} d\omega.$$

3. To describe the structure of the fundamental solution (5), we expand $e^{(\xi-\tau)\lambda}$ in a series and successively bring, in each term, the operator Δ_B under the integral sign and then integrate by parts. Then, for the case when γ is not even, we obtain

$$K(x) = r^{2m-\gamma} \sum_{i=0}^{\infty} r^i \Omega_i \left(\frac{x}{r} \right),$$

where $\Omega_i(x/r)$ are infinitely differentiable functions, even with respect to the variable x_{n+1} . In the case of even γ we have

$$K(x) = r^{2m-\gamma} \sum_{i=0}^{\infty} r^i \Omega_i \left(\frac{x}{r} \right) + W(x) \ln r,$$

where $W(x)$ is a regular solution of the equation $\mathcal{L}(D_x, B_{x_{n+1}})W = 0$. In the case of a homogeneous operator we have

$$K(x) = r^{2m-\gamma} \Omega \left(\frac{x}{r} \right), \quad (6)$$

$$K(x) = r^{2m-\gamma} \Omega \left(\frac{x}{r} \right) + q_{2m-\gamma}(x) \ln r, \quad (7)$$

where $q_{2m-\gamma}(x)$ is a homogeneous polynomial of degree $2m - \gamma$, even in x_{n+1} . From formulas (6) and (7) we obtain the estimates

$$\left| D_x^i B_{x_{n+1}}^j K(x) \right| \leq \text{const} \cdot r^{2m-\gamma-i-2j}.$$

If γ is even and $i + 2j \leq 2m - \gamma$, then

$$\left| D_x^i B_{x_{n+1}}^j K(x) \right| \leq \text{const} \cdot r^{2m-\gamma-i-2j}(1 + |\ln r|).$$

Inside the domain D ($x_{n+1} > 0$), the fundamental solution and its derivatives have the same singularity as the fundamental solutions of ordinary elliptic equations.

4. We now consider the operator $\mathcal{L}(D_x, B_{x_{n+1}})$, whose coefficients $A_{j_1, \dots, j_i}^\nu(x)$ are i times continuously differentiable in the domain D with respect to the arguments $x_1, \dots, x_n$ and withstand ν applications of the operator $B_{x_{n+1}}$. Let $u(x)$ and $v(x)$ be two functions, $2m$ times continuously differentiable in the domain D with respect to the arguments $x_1, \dots, x_n$ and withstanding m applications of the operator $B_{x_{n+1}}$. We denote the class of such functions by C_B^{2m} . Then the following Green formula is valid:

$$\int_D (v \mathcal{L}u - u \bar{\mathcal{L}}v) x_{n+1}^k dx = \int_S R[u, v] x_{n+1}^k dS, \quad (8)$$

where

$$\bar{\mathcal{L}}v = \sum_{\nu=0}^m \sum_{i=0}^{2m-2\nu} (-1)^i \sum_{j_1, \dots, j_i=1}^n \frac{\partial^j}{\partial x_{j_1} \dots \partial x_{j_i}} B^\nu (A_{j_1, \dots, j_i}^\nu(x) v),$$

and $R[u, v]$ is a bilinear differential operator.

For any point $z \in D$ we define the fundamental solution $K(x, z)$ with pole at the point z as such a function of x that the identity

$$v(z) = \int_D \bar{\mathcal{L}}(v) K(x, z) x_{n+1}^k dx + \int_S R[K(x, z), v(x)] x_{n+1}^k dS \quad (9)$$

holds, where $v(x) \in C_B^{2m}$. Following the Levi method ⁽⁵⁾, we introduce into consideration the operator $\mathcal{L}^z$, a homogeneous operator with constant coefficients, which is obtained from the operator $\mathcal{L}(D_x, B_{x_{n+1}})$ by retaining the highest-order terms and freezing the coefficients at the point z . Let $K^+(x, z)$ be the fundamental solution of the operator $\mathcal{L}^z$ with pole at the point z . We shall seek the function $K(x, z)$ in the form

$$K(x, z) = K^+(x, z) + \int_D K^+(x, y) u(y, z) y_{n+1}^k dy. \quad (10)$$

Using Green's formula (8) and identity (9), with respect to the function $u(y, z)$, we obtain an integral equation with a kernel having a weak singularity, whence it follows that, for a sufficiently small domain D , there exists a unique solution of this equation. Then formula (10) gives the desired fundamental solution $K(x, z)$ of the operator $\mathcal{L}(D_x, B)$.

In conclusion, I express my sincere gratitude to I. A. Kipriyanov for posing the problem and for his constant assistance in the work.

Received
10 III 1966

REFERENCES

1. I. A. Kipriyanov, DAN, **158**, No. 2 (1964).
2. I. A. Kipriyanov, V. I. Kononenko, DAN, **170**, No. 2 (1966).
3. B. M. Levitan, UMN, **6**, 2 (42) (1951).
4. F. John, *Plane Waves and Spherical Means Applied to Partial Differential Equations*, Moscow, 1958.
5. E. E. Levi, UMN, **8**, 249 (1940).
6. A. Weinstein, Proc. Simp. Univ. of Maryland, 1961.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.