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1967

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Abstract

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UDC 517.946.4

MATHEMATICS

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ON A SINGULAR PROBLEM WITH CONDITIONS OF F. I. FRANKL AND F. TRICOMI

(Presented by Academician I. N. Vekua, 21 VI 1966)

1. Consider the equation

$$S(u) \equiv u_{xx} + \operatorname{sgn} y u_{yy} + \frac{2q}{x} u_x = 0, \quad q > 1 \quad (\text{S})$$

in the domain D , bounded by: 1) the arc AB of the circle $x^2 + y^2 = 1$, $A(1; 0)$, $B(0; 1)$; 2) the segment OB of the axis OY ; 3) the segment OC of the characteristic $x + y = 0$, $C(c; -c)$, $c \geq 1/2$; 4) the chordal arc AC , which either coincides completely with the segment CI ($I(2c; 0)$) of the characteristic $x - y = 2c$, or is situated between this characteristic and the segment AE ($E(1/2; -1/2)$) of the characteristic $x - y = 1$. The arc AC is such that any characteristic can intersect it in no more than one point. Let D_1 and D_2 be, respectively, the subdomains of ellipticity and hyperbolicity of equation (S). The magnitude of the arc s along the boundary of the domain D will be measured from the point A counterclockwise. Let l be the length of AC , and TR a segment of the arc AB , whose length is equal to the length of AC and whose endpoints T and R have respectively the arc coordinates $\varepsilon > 0$, $\varepsilon + l < \pi/2$.

Singular problem $\tilde{A}_1$. In the domain D , find a function $u(x, y)$ possessing the properties: $S(u) = 0$ in $D_1 \cup D_2$, $u \in C(\bar{D})$, $u \in C^1(D \setminus OA)$,

$$u_y(x; 0-0) = u_y(x; 0+0)$$

(the "gluing" condition), and satisfying the following boundary conditions:

$$\begin{aligned} 1. \quad u|_{\overline{AT}} &= \gamma_1(s). & 3. \quad u|_{\overline{OC}} &= \psi(x). \\ 2. \quad u|_{\overline{RB}} &= \gamma_2(s). & 4. \quad u(\varepsilon + \mu) - u(-\mu) &= g(\mu), \quad 0 \leq \mu \leq l. \end{aligned} \quad (\text{I})$$

Here $\gamma_1, \gamma_2, \psi, g$ are prescribed functions. Condition 4 defines a curvilinear jump of compactification¹⁻³.

A **generalized solution** of problem $\tilde{A}_1$ of class $\tilde{\mathcal{G}}_2$ (of class $\tilde{\mathcal{G}}_2^0$) will mean a function $u(x, y)$, defined in D and satisfying the conditions: 1) $S(u) = 0$ in D_1 ; 2) $u \in C^2(D_1)$; 3) $u \in C(\bar{D})$; 4) u is a generalized solution of equation (S) of class $\tilde{\mathcal{G}}_2$ (class $\tilde{\mathcal{G}}_2^0$) in the domain D_2 (such classes of solutions are introduced below); 5) for any x , $0 < x < 1$, there exists

$$\lim_{y \rightarrow 0} u_y = v(x),$$

where $v(x)$ is absolutely integrable on $0 \leq x \leq 1$ and satisfies the Hölder condition on $0 < x < 1$; 6) u satisfies the boundary conditions (I).

2. Problem NE. In the domain D_1 , find a bounded solution of equation (S) from the data:

$$\lim_{y \rightarrow 0} u_y = v(x), \quad 0 < x < 1; \quad u|_{AB} = \varphi(s). \quad (\text{II})$$

Without loss of generality one may assume

$$\gamma_1(0) = \gamma_2(\pi/2) = \psi(0) = 0.$$

In the domain D_1 , the generalized solution of problem $\tilde{A}_1$ of class $\tilde{\mathcal{G}}_2$ is determined by the formula ⁽⁴⁾

$$u(x_0, y_0) = - \int_0^1 G_0(t, 0; x_0, y_0) v(t) dt - \int_0^{\pi/2} \frac{\partial G_0}{\partial n} \varphi(s) ds, \quad (1)$$

where $G_0(x, y; x_0, y_0)$ is the Green function of problem NE.

Lemma 1. Let $u(x, y)$ be a generalized solution of problem $\tilde{A}_1$ of class $\tilde{\mathcal{G}}_2$. Then $u(x, y)$ cannot attain in $\bar{D}_1$ a greatest positive (least negative) value on the interval $x = 0$, $0 < y < 1$, greater (smaller) than the values of $u(x, y)$ on $\overline{AB \cup AO}$.

3. Problem $\tilde{\mathcal{G}}_2$. In the domain D_2 , find a solution $u(x, y)$ of equation (S) from the data

$$\lim_{y \rightarrow 0} u_y = \nu(x), \quad 0 < x < 1; \quad u|_{OC} = \psi(x). \quad (\text{III})$$

By the change of variables $\xi = y + x$, $\eta = y - x$ in the domain D_2 , equation (S) is transformed to the form

$$S_0(u) \equiv u_{\xi\eta} - \frac{q}{\xi - \eta} (u_\xi - u_\eta) = 0.$$

The domain D_2 goes over into the triangle $O'A'C'$ (the domain Δ). The boundary conditions (III) take the form

$$\lim_{(\xi, \eta) \rightarrow (x; -x)} (u_\xi + u_\eta) = \nu(x), \quad 0 < x < 1; \quad u(0; \eta) = \psi(-\eta/2) = \psi_1(\eta).$$

We shall call the function $u(\xi, \eta)$ a **regular solution** of equation (S_0) in Δ (of equation (S) in D_2) if $S(u) = 0$ in Δ , $u \in C^2(\Delta)$, $u \in C(\bar{\Delta})$, $u \in C^1(\bar{\Delta} \setminus (O' \cup A'))$.

Lemma 2. If the function $\nu(\xi)$ is continuously differentiable on $0 < \xi < 1$ and absolutely integrable on $0 \leq \xi \leq 1$, the function $\psi_1(\eta)$ is continuous on $-2c \leq \eta \leq 0$, twice continuously differentiable on $(-2c, 0)$, and $\psi_1(\eta) = (-\eta)^\alpha \tilde{\psi}(\eta)$ ($0 < \alpha \leq 1$, $\tilde{\psi}(\eta)$ bounded), then the regular solution $u(\xi_0, \eta_0)$ of equation (S_0) is determined by the formula

$$\begin{aligned} u(\xi_0, \eta_0) = & \int_0^{\xi_0} H(\xi, \xi_0, \eta_0) \nu(\xi) d\xi - \int_{\eta_0}^0 \left[\frac{\partial A(0, \eta; \xi_0, \eta_0)}{\partial \eta} - \frac{q}{\eta} A(0, \eta; \xi, \eta_0) \right] \times \\ & \times \psi_1(\eta) d\eta + \left(\frac{\xi_0}{\xi_0 - \eta_0} \right)^q \psi_1(-\xi_0) + \left(\frac{-\eta_0}{\xi_0 - \eta_0} \right)^q \psi_1(\eta_0). \end{aligned} \quad (2)$$

Here $H(\xi, \xi_0, \eta_0)$ is the value for $\xi = -\eta$ of the Riemann function; $A(\xi, \eta; \xi_0, \eta_0)$ is the Riemann-Hadamard function of problem CG ⁽⁴⁾. The proof is analogous to ⁽⁵⁾, Theorem 6.

By a **generalized solution of class $\tilde{\mathcal{G}}_2$** of equation (S) in the domain D_2 we shall mean the function defined by formula (2), in which $\nu(\xi)$ is continuous on $0 < \xi < 1$, $\psi_1(\eta)$ is continuous on $-2c \leq \eta \leq 0$, and

$$\psi_1(\eta) = (-\eta)^\alpha \tilde{\psi}(\eta) \quad (0 < \alpha < 1, \tilde{\psi}(\eta) \text{ bounded}).$$

If, in addition, $\psi'(0) = 0$, $\nu(\xi)$ and $\psi'_1(\eta)$ satisfy a Hölder condition respectively on $0 < \xi < 1$ and $-2c \leq \eta \leq 0$, then the function $u(\xi_0, \eta_0)$ will be regarded as belonging to the class $\tilde{\mathcal{G}}_2^0$ of generalized solutions of equation (S) in D_2 .

Lemma 3. Every generalized solution of class $\tilde{\mathcal{G}}_2$ of equation (S) in the domain D_2 can be represented as the limit of a sequence, uniformly convergent in $\bar{D}_2$, of regular solutions of equation (S) in D_2 .

Lemma 4. Let $u(\xi, \eta)$ be a regular solution of equation (S_0) , with $u(0, \eta) = 0$. Then the maximum of $u(\xi, \eta)$ in $\bar{\Delta}$, if it is positive, is attained only on the segment $O'A'$.

The proof follows from $q/(\xi - \eta) > 0$, $(q - q^2)/(\xi - \eta) < 0$ in Δ .

Lemma 5. Let $u(\xi, \eta)$ be a generalized solution of class $\tilde{\mathcal{G}}_2$ of equation (S) in D_2 , with $u(0, \eta) = 0$. Then the maximum of $u(\xi, \eta)$ in $\bar{\Delta}$, if it is positive, is attained on the segment $O'A'$.

The proof follows from Lemmas 3 and 4.

4. Theorem 1. There can exist no more than one generalized solution of problem $\tilde{A}_1$ of class $\tilde{\mathcal{G}}_2$.

Suppose the contrary. Then there exists a generalized solution $u(x, y)$ of problem $\tilde{A}_1$ of class $\tilde{\mathcal{G}}_2$, taking positive values, satisfying

satisfying homogeneous boundary conditions. Suppose the maximum of $u(x, y)$ in $\bar{D}$ is attained in $\bar{D}_2$. Then, by Lemma 5, it is attained at an interior point of the segment OA , which contradicts the gluing condition. Assuming that the maximum of $u(x, y)$ in $\bar{D}$ is attained in $\bar{D}_1$, by virtue of the preceding, Lemma 1 and the density jump condition we obtain that it is attained in $\bar{D}_2$, which is impossible.

5. Solution of problem $\tilde{A}_1$. Let $u(x, y)$ be a generalized solution of problem $\tilde{A}_1$ of the class $\mathcal{G}_2^0$. Taking into account (1), (2) and the gluing condition, we obtain an integral equation of the first kind with unknown function $\nu(x)$. Differentiating and transforming this equation, we arrive at the singular integral equation

$$\nu(x) + \frac{1}{\pi} \int_0^1 \left[\frac{1}{t-x} - \frac{1}{1-tx} + \frac{\mathcal{L}(t, x)}{t+x} \right] \nu(t) dt = \int_0^1 R(t, x) \nu'(t) dt + f(x), \quad (3)$$

where $\mathcal{L}(t, x)$ is a homogeneous function, not depending, like $R(t, x)$, on the boundary conditions. The kernel $R(t, x)$ is continuous everywhere in the square $0 \leq t, x \leq 1$, with the exception of the point $(1; 1)$, at which it has a singularity of arbitrarily small order. The function $f(x)$ depends on $\gamma_1(s)$, $\gamma_2(s)$, $\psi(x)$, $g(\mu)$, $\nu(x)$.

Lemma 6. In equation (3) the function $f(x)$ can be represented in the form

$$f(x) = f_0(x) + \int_0^1 Z(t, x) \nu(t) dt, \quad (4)$$

where $f_0(x)$ does not depend on $\nu(x)$, and the kernel $Z(t, x)$ is continuous in the square $0 \leq t, x \leq 1$.

Taking Lemma 6 into account, we write equation (3) in the form

$$\nu(x) + \frac{1}{\pi} \int_0^1 \left[\frac{1}{t-x} - \frac{1}{1-tx} + \frac{\mathcal{L}(t,x)}{t+x} \right] \nu(t) dt = \int_0^1 R_0(t,x) \nu(t) dt + f_0(x), \quad (5)$$

where $R_0(t, x) = R(t, x) + Z(t, x)$.

Lemma 7. If the function $\gamma_1(s)$ has a derivative of second order at least in a neighborhood of the point $s = 0$, then the function $f_0(x)$ satisfies the Hölder condition on $0 \leq x \leq 1$.

Applying the method of work (6), we reduce equation (5) to the equivalent Fredholm equation

$$\varphi(\xi) = \int_0^\infty R_{00}(\eta, \xi) \varphi(\eta) d\eta + f_{00}(\xi), \quad (6)$$

$$\varphi(\xi) = e^{-\xi} \nu(e^{-2\xi}), \quad x = e^{-2\xi}, \quad t = e^{-2\eta}. \quad (7)$$

The function $f_{00}(\xi) \equiv 0$ if $\gamma_1(s) = \gamma_2(s) = \psi(x) = g(\mu) \equiv 0$. From Theorem 1 follows the uniqueness and existence of a solution of equation (6) in the class of functions satisfying the Hölder condition and absolutely integrable with some power $p > 1$ on $0 < \xi < +\infty$. Having found $\varphi(\xi)$ from (6), we find $\nu(x)$ from (7). By formula (2) we find $u(x, y)$ in $\overline{D}_2$, and, consequently, also $u|_{\overline{AC}}$. With the help of the density jump condition we find $u|_{\overline{TR}}$, and then by formula (1) determine $u(x, y)$ in $\overline{D}_1$. Thus, the following is valid.

Theorem 2. If the function $\gamma_1(s)$ has a derivative of second order at least in a neighborhood of the point $s = 0$, then there exists a unique generalized solution of problem $\tilde{A}_1$ of the class $\mathcal{G}_2^0$.

Received
4 VI 1966

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