

NONLOCAL SOLVABILITY OF BOUNDARY-VALUE PROBLEMS FOR QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS OF HYPERBOLIC TYPE

S. Ya. Yakubov

1967

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196701.70291>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.947.35

MATHEMATICS

S. Ya. YAKUBOV

NONLOCAL SOLVABILITY OF BOUNDARY-VALUE PROBLEMS FOR QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS OF HYPERBOLIC TYPE

(Presented by Academician I. G. Petrovskii on 8 II 1967)

Consider, in a bounded domain Ω of Euclidean space R^n , formally self-adjoint elliptic operators of order $2m$

$$A(t, x, D)u(x) = \sum_{|\alpha|, |\beta| \leq m} D^\alpha (a_{\alpha, \beta}(t, x) D^\beta u(x)) \quad (t \in [0, T]) \quad (1)$$

with homogeneous boundary conditions

$$B_j(x, D)u(x)|_\Gamma \equiv \sum_{|\alpha| \leq k_j} b_\alpha^j(x) D^\alpha u(x)|_\Gamma = 0 \quad (j = 1, \dots, m), \quad (2)$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$; $|\alpha| = \alpha_1 + \dots + \alpha_n$; $D^\alpha = \partial^{|\alpha|} / \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$; the functions $a_{\alpha, \beta}(t, x)$ and $b_\alpha^j(x)$ are real; $a_{\alpha, \beta}(t, x) = (-1)^{|\alpha| + |\beta|} \cdot a_{\beta, \alpha}(t, x)$; $k_j \leq 2m - 1$, and

$$\sum_{|\alpha| = |\beta| = m} a_{\alpha, \beta}(t, x) \xi^\alpha \xi^\beta \geq a \sum_{i=1}^n \xi_i^{2m}, \quad a = \text{const} > 0,$$

for every real $\xi = (\xi_1, \dots, \xi_n)$; $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$. Let the boundary Γ of the domain Ω be continuously differentiable $2m$ times. Let the boundary conditions (2) cover the differential expression (1)¹ (the Shapiro-Lopatinskii condition).

Let the elliptic operator $A(t)^{(1,2)}$, generated by the system $\{A(t, x, D), B_j(x, D)\}$, be a self-adjoint operator in $L_2(\Omega)$. (This is a condition on the boundary operators B_j . For example, if $B_j(x, D) = \partial^j / \partial n^j$ and $j = 0, \dots, m - 1$ (the Dirichlet problem) or $j = m, \dots, 2m - 1$ (the Neumann problem), then this condition is satisfied.) As is known⁽²⁾, the domain of definition of the elliptic operator $A(t)$ is a closed subspace of $W_2^{2m}(\Omega)$.

By $C(t, x, D)$ we denote any differential operator of order not exceeding m :

$$C(t, x, D) = \sum_{|\alpha| \leq m} C_\alpha(t, x) D^\alpha u(x).$$

In the article the following problems are investigated:

I. For $n < 2m$, find a function $u(t, x)$, defined in the cylinder $Q[0, T] \times \Omega$, satisfying the equation

$$\begin{aligned} \frac{\partial^2 u(t, x)}{\partial t^2} + A(t, x, D)u(t, x) + C(t, x, D)u(t, x) = \\ = F(t, x, u, \dots, D^\delta u, |u|^2, \dots, |D^\delta u|^2) \frac{\partial u(t, x)}{\partial t} + \frac{d\Phi(|u|^2)}{dr} u + \tilde{f}(t, x), \end{aligned} \quad (3)$$

the initial conditions

$$u(t, x)|_{t=0} = u_0(x), \quad \partial u(t, x)/\partial t|_{t=0} = u_1(x) \quad (4)$$

and the boundary conditions (2).

II. For $2m \leq n < 4m$, find a function $u(t, x)$ satisfying the equation

$$\frac{\partial^2 u(t, x)}{\partial t^2} + A(t, x, D)u(t, x) + C(t, x, D)u(t, x) = d\Phi(|\nu(t, x)|^2) / d\tau u(t, x) + f(t, x) \quad (5)$$

and the conditions (2), (4).

Problems I and II reduce to the abstract Cauchy problem

$$\frac{d^2 u(t)}{dt^2} + A(t)u(t) = f(t, u(t), du(t)/dt), \quad u(0) = u_0, \quad u'(0) = u_1 \quad (6)$$

in the Hilbert space $L_2(\Omega)$. By $L_2(\Omega)$ we denote the space of complex-valued functions summable with the square of the modulus. Problem (6) in a Hilbert space H was investigated in the author's paper ⁽³⁾. Particular cases of problem (6) were investigated in papers ⁽⁴⁻⁶⁾. (The results obtained in ⁽³⁾, even for these particular cases, turned out to be new.) In paper ⁽⁷⁾ problem (6) was investigated under other assumptions. Problem (6) was also investigated in a Banach space in the author's papers ^(8, 9). We state one theorem from ⁽³⁾, by means of which the solvability of problems I and II as a whole is established.

Theorem 1. *Suppose that the following conditions are satisfied:*

1°. For each $t \in [0, T]$ the operator $A(t)$ is self-adjoint and positive definite in H , and for all $t \in [0, T]$ and $u \in D(A(t))$ the inequality

$$(A(t)u, u) \geq \gamma(u, u) \quad (\gamma = \text{const} > 0).$$

holds.

2°. The operator $A(t)$ has a domain of definition $D(A)$ independent of t ; the operator-function $A(t)A^{-1}(0)$ is strongly continuously differentiable.

3°. The operator $f(t, u, v)$ maps $[0, T] \times H(A) \times H(A^{1/2})$ into H and is bounded; for every function $u(t)$ continuous in $H(A)$, continuously differentiable in $H(A^{1/2})$, and every function $v(t)$ continuous in $H(A^{1/2})$, continuously differentiable in H , the vector-function $f(t, u(t), v(t))$ is continuously differentiable in H ; for any two pairs of functions $u_1(t), u_2(t), v_1(t), v_2(t)$, respectively satisfying these smoothness conditions, the inequality

$$\begin{aligned} & \left\| \frac{d}{dt} [f(t, u_1(t), v_1(t)) - f(t, u_2(t), v_2(t))] \right\| \leq \\ & \leq C(R) [\|A(0)(u_1(t) - u_2(t))\| + \|A^{1/2}(0)(u'_1(t) - u'_2(t))\| + \\ & + \|A^{1/2}(0)(v_1(t) - v_2(t))\| + \|v'_1(t) - v'_2(t)\|], \end{aligned}$$

as soon as

$$\|A(0)u_i(t)\| + \|A^{1/2}(0)u'_i(t)\| \leq R \quad \text{and} \quad \|A^{1/2}(0)v_i(t)\| + \|v'_i(t)\| \leq R \quad (i = 1, 2).$$

4°. For any functions $u(t)$ continuous in $H(A)$, continuously differentiable in $H(A^{1/2})$, twice continuously differentiable in H , one has

$$\text{Re} \int_0^t \left(f \left(\tau, u(\tau), \frac{du(\tau)}{d\tau} \right), \frac{du(\tau)}{d\tau} \right) d\tau \leq C \left[1 + \int_0^t \left(\|A^{1/2}(0)u(\tau)\|^2 + \left\| \frac{du(\tau)}{d\tau} \right\|^2 \right) d\tau \right];$$

for all functions $u(t)$ satisfying these smoothness conditions, the inequality

$$\left\| \frac{d}{dt} f(t, u(t), u'(t)) \right\| \leq C(R) \left[1 + \|A(0)u(t)\| + \left\| A^{1/2}(0) \frac{du(t)}{dt} \right\| + \left\| \frac{d^2 u(t)}{dt^2} \right\| \right], \quad (7)$$

holds as soon as

$$\|A^{1/2}(0)u(t)\| + \|du(t)/dt\| \leq R.$$

5°. $u_0 \in D(A)$, $u_1 \in D(A^{1/2})$.

Then problem (6) has a unique solution on $[0, T]$.

* By $H(A^\alpha)$ is denoted the Hilbert space consisting of elements of $D(A^\alpha)$ with scalar product $(u, v)_{A^\alpha} = (A^\alpha(0)u, A^\alpha(0)v)$.

This theorem makes it possible to establish the following theorems.

Theorem 2. *Let the following conditions be satisfied:*

1°. $a_{\alpha, \beta}(t, x) \in C^{(\alpha)}(\bar{\Omega})$, $b_\alpha^i \in C^{2m-k_j}$; the functions $D^l a_{\alpha, \beta}(t, x)$ ($|l| \leq |\alpha|$), $C_\alpha(t, x)$, and their first derivatives with respect to t are continuous in $\bar{Q}$.

2°. $|\delta| \leq (2m - n)/4$; the function $F(t, x, z_s, r_s)$ is continuous together with its derivatives with respect to t, z_s, r_s in the domain $\{t \in [0, T], x \in \bar{\Omega}, |z_s| < \infty, 0 \leq r_s < \infty\}$; the function $\Phi(r)$ is twice continuously differentiable on $[0, \infty)$; the function $f(t, x)$ is continuous together with its derivative with respect to t in the domain $\{t \in [0, T], x \in \bar{\Omega}\}$; the functions $\partial F/\partial t$, $\partial F/\partial z_s$, $\partial F/\partial r_s$, and $d^2\Phi/dr^2$ satisfy, in z_s, r_s , the Lipschitz condition in every bounded set $|z_s| \leq R$, $0 \leq r_s \leq R$, with Lipschitz constants depending only on R .

3°. $\operatorname{Re} F(t, x, z_s, r_s) \leq C$, $\operatorname{Re} \Phi(r) \leq C(1 + r)$.

4°. $u_0(x) \in W_2^{2m}(\Omega, B)$, $u_1(x) \in W_2^m(\Omega, B)$.

Then problem I has a unique solution in the cylinder $Q = [0, T] \times \Omega$.

Theorem 3. *Let conditions 1° and 4° of Theorem 2 be satisfied. Suppose further:*

5°. The function $f(t, x)$, together with its derivative $\partial f(t, x)/\partial t$, is continuous in $\bar{Q}$; the function $\Phi(r)$ is three times continuously differentiable on $[0, \infty)$; the following estimates hold:*

$$\operatorname{Re} \Phi(r) \leq C(1 + r),$$

$$|\Phi'(r)| \leq C(1 + r^q), \quad |\Phi''(r)| \leq C(1 + r^{q-1}), \quad |\Phi'''(r)| \leq C(1 + r^{q-2}),$$

where $q = 2m(n - 2m)^{-1}$ for $n > 2m$; q is arbitrary for $n = 2m$.

Then problem II has a unique solution in the cylinder $Q = [0, T] \times \Omega$.

Theorems 2 and 3 improve and generalize the results of the papers ^(4,5). For example, in ⁽⁴⁾, for $n > 2m$, it is assumed that $q = m(n - 2m)^{-1}$. Similar results were obtained in the article ⁽¹⁰⁾.

We indicate the plan of the proof.

Consider the nonlinear operator

$$f(t, u, v) = -C(t, x, D)u(x) + \lambda u(x) + \\ + F(t, x, u, \dots, D^\delta u, |u|^2, \dots, |D^\delta u|^2)v + \frac{d\Phi(|u|^2)}{dr}u + f(t, x)$$

in the space $L_2 \times L_2$. For lack of space, we study the “principal” term $F(u) = \Phi'(|u|^2)u$ of the operator $f(t, u, v)$. Since the main restrictions and difficulties arise in establishing the weakened two-sided estimate (7), for the operator $F(u)$, when $n > 2m$, we verify inequality (7). From

$$\frac{d}{dt}F(u(t)) = \Phi'(|u(t)|^2)u'(t) + \Phi''(|u(t)|^2)[|u(t)|^2u'(t) + u^2(t)\overline{(u(t))'}]$$

it follows that

$$\left\| \frac{d}{dt}F(u(t)) \right\|_2^2 \leq C \int_{\Omega} |u(t, x)|^{4q} \left| \frac{\partial u(t, x)}{\partial t} \right|^2 dx.$$

Since $u(t, x) \in W_2^{2m}$, $\partial u(t, x)/\partial t \in W_2^m$, and $\|u(t)\|_{W_2^{2m}} + \|\partial u(t)/\partial t\|_2 \leq R$, then for $2m < n < 4m$, applying the results of the article ⁽¹¹⁾, we obtain the inequality

$$\left\| \frac{d}{dt}F(u(t)) \right\|_2^2 \leq C \|u(t)\|_C^{4q} \left\| \frac{du(t)}{dt} \right\|_2^2 \leq C(R) \|u(t)\|_C^{4q} \leq \\ \leq C(R) (\|u(t)\|_{W_p^l}^\tau \|u(t)\|_p^{1-\tau})^{4q} \leq C(R) \|u(t)\|_{W_p^l}^{4q\tau},$$

where $p = 2n/(n - 2m)$, $l > (n - 2m)/2$, $\tau = (n - 2m)/2l$. On the other hand, by virtue of ^(12,13), for any $0 \leq l \leq 2m$, the inequality

$$\|u\|_{W_p^l} \leq C \|u\|_{W_2^{2m}}^{l/2m} \|u\|_p^{(2m-l)/2m}.$$

Thus, for any $l \in ((n - 2m)/2, 2m]$ we have

$$\left\| \frac{d}{dt}F(u(t)) \right\|_2^2 \leq C(R) \|u(t)\|_{W_2^{2m}}^{\frac{l}{2m}4q\tau} \leq C(R) \|u(t)\|_{W_2^{2m}}^2,$$

whence, in turn, inequality (7) follows.

Remark. The results obtained carry over to quasilinear hyperbolic systems with a strongly elliptic principal part.

Let us note that, for the equation

$$\begin{aligned} \partial^2 u / \partial t^2 - a \Delta u + b(t) u^p &= 0, & \Delta &= \partial^2 / \partial x_1^2 + \partial^2 / \partial x_2^2 + \partial^2 / \partial x_3^2, \\ b(t) &\geq 0, \end{aligned} \quad (9)$$

which occurs in quantum field theory, the results of papers (4-7, 10) are applicable only for $p = 1$ and $p = 3$. As is easy to see, Theorem 3 is applicable to equation (9) also for $p = 5$.

I express my deep gratitude to Prof. S. G. Krein for his constant attention and for discussion of the results obtained.

Institute of Mathematics and Mechanics
Academy of Sciences of the Azerbaijan SSR

Received
7 II 1967

REFERENCES

1. S. Agmon, A. Douglis, L. Nirenberg, *Comm. on Pure and Appl. Math.*, **12**, 623 (1959).
2. S. Agmon, *Comm. on Pure and Appl. Math.*, **15**, 119 (1962).
3. S. N. Yakubov, *DAN*, **176**, No. 1 (1967).
4. F. E. Browder, *Math. Zs.*, **80**, No. 3 (1962).
5. F. E. Browder, W. A. Strauss, *Pacif. J. Math.*, **13**, No. 1 (1963).
6. Ya. D. Mamedov, *DAN*, **158**, No. 1 (1964).
7. V. A. Pogorelenko, P. E. Sobolevskii, *Ukr. Mat. Zhurn.*, No. 1 (1967).
8. S. Ya. Yakubov, *Dokl. AN AzerbSSR*, **20**, No. 4 (1964).
9. S. Ya. Yakubov, *DAN*, **168**, No. 4 (1966).
10. J. Lions, W. A. Strauss, *Bull. Soc. Math. France*, **93**, No. 1, 43 (1965).
11. V. P. Glushko, S. G. Krein, *DAN*, **122**, No. 6 (1958).

12. E. Gagliardo, *Ricerche mat.*, **8**, No. 1, 24 (1959).

13. L. Nirenberg, *Ann. Sc. Norm. Sup. Pisa*, ser. 3, **13**, 2, 115 (1959).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.