

ON LOCALLY PERFECT MAPPINGS

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1967

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Abstract

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UDC 513.83

MATHEMATICS

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ON LOCALLY PERFECT MAPPINGS

(Presented by Academician P. S. Aleksandrov on 10 VIII 1966)

This paper considers locally perfect mappings, which stand in the same relation to perfect mappings as locally bicomact spaces do to bicomacta. All spaces are assumed to be T_1 -spaces, and all mappings continuous.

A mapping $f : X \rightarrow Y$ will be called **locally perfect** if for each point x there exists a neighborhood U such that the image $f[U]$ of the closure U is closed in Y and $f|U$ is perfect.

Every mapping of a locally bicomact space is locally perfect. In particular, every mapping of a discrete space is locally perfect. Also, if $f : X \rightarrow Y$ is perfect and X_0 is an open set in X , then the restriction $f|X_0$ is locally perfect.

Let $f : X \rightarrow Y$ be locally perfect. Denote by Y^* the set of points $y \in Y$ at which f is not perfect (i.e., either f is not closed, or the full preimage $f^{-1}y$ is not bicomact).

Theorem 1. *Let $f : X \rightarrow Y$ be a locally perfect mapping. Then there exist a space $\tilde{X} \supset X$ and a perfect mapping $\tilde{f} : \tilde{X} \rightarrow Y$ such that: 1) $\tilde{f}|X = f$; 2) $\tilde{f}(\tilde{X} \setminus X) = Y^*$; 3) the restriction $\tilde{f}|(\tilde{X} \setminus X)$ is one-to-one; 4) X is dense in $\tilde{X}$; 5) X is open in $\tilde{X}$. These conditions determine $\tilde{X}$ and $\tilde{f}$ uniquely.*

The space $\tilde{X}$ is obtained from X as follows: to each full preimage of a point $y \in Y^*$ we add one new point, and to the full preimages of points $y \in Y \setminus Y^*$ we add nothing. We leave the topology on X unchanged. If $x \in \tilde{X} \setminus X$, then its neighborhoods will be sets of the form $U = \tilde{f}^{-1}V \setminus F$, where $V = V_y$ is an arbitrary neighborhood of the point $y = \tilde{f}x$ in Y , $F \subset X$ and is closed in X , fF is closed in Y , and the mapping $f|F$ is perfect.

The mapping $\tilde{f}$ is closed at the points $y \in Y^*$. Indeed, let Q be open in $\tilde{X}$, $\tilde{f}^{-1}y \subset Q$. There exists a neighborhood $U = \tilde{f}^{-1}V \setminus F$ of the point $x_0 = \tilde{f}^{-1}y \cap (\tilde{X} \setminus X)$, contained in Q . The set $F_1 = F \cap (Q \cap X)$ is closed in F , hence fF_1 is closed in Y , and $y \notin fF_1$. Therefore $V_1 = V \setminus fF_1$ is a neighborhood of the point y . It is not hard to see that $\tilde{f}^{-1}V_1 \subset Q$. The closedness of $\tilde{f}$ on $Y \setminus Y^*$ follows from the openness of this set in Y . It is easy to verify that the full preimages $\tilde{f}^{-1}y$, where $y \in Y^*$, are bicomact. Thus the mapping $\tilde{f}$ is perfect.

In what follows, for a locally perfect mapping $f : X \rightarrow Y$, the symbols $\widetilde{X}$ and $\widetilde{f}$ will always denote the space and mapping discussed in the theorem.

Let, for example, $X = \{(x, y) : 0 \leq x \leq 1, 0 \leq y < 1\}$, Y be the segment $[0, 1]$ of the abscissa axis, and let f be the vertical projection of X onto Y . The mapping f is locally perfect, although it is not closed at any point of the space Y . In this case

$$\widetilde{X} = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

and $\widetilde{f}$ is the vertical projection onto Y .

Theorem 2. *Let $f : X \rightarrow Y$ be a locally perfect mapping. If Y is locally bicomact at the point y_0 , then X is locally bicomact at each point $x \in f^{-1}y_0$. Consequently, if Y is a locally bicomact space, then X is also locally bicomact.*

However, under passage to the image under a locally perfect mapping, local bicomactness is not preserved.

Theorem 3. Let the mapping $f : X \rightarrow Y$ be locally perfect.

1) If X and Y are Hausdorff, then $\widetilde{X}$ is Hausdorff; 2) if X and Y are completely regular, then $\widetilde{X}$ is completely regular; 3) if X and Y have countable bases, then $\widetilde{X}$ has a countable base.

1) This is proved directly.

2) Let B be closed in $\widetilde{X}$ and $x_0 \notin B$, $\widetilde{f}x_0 \notin \widetilde{f}B$. Then the existence of a function continuous on $\widetilde{X}$, equal to zero at the point x_0 and to one on B , follows from the complete regularity of Y . If $y_0 = \widetilde{f}x_0 \in \widetilde{f}B$, $x_0 \in \widetilde{X} \setminus X$, then, since X is completely regular and the set $A = B \cap \widetilde{f}^{-1}y_0$ is bicomact, there exists a neighborhood U , $[U]_{\widetilde{X}} \subset X$, of the set A and a function ψ , continuous on X , such that $\psi|_{X \setminus U} \equiv 0$ and $\psi \equiv 1$ in some smaller neighborhood W , $[W]_X \subset U$, of the set A . Then the function φ_1 , $\varphi_1(x) = \psi(x)$ if $x \in U$, and $\varphi_1(x) = 0$ if $x \in \widetilde{X} \setminus U$, is continuous on $\widetilde{X}$, $\varphi_1(x_0) = 0$, and $\varphi_1|_W \equiv 1$. Since $B_1 = B \setminus W$ is closed in $\widetilde{X}$, and $y_0 \notin \widetilde{f}B_1$, by what has been proved there exists a continuous function φ_2 , $\varphi_2(x_0) = 0$, $\varphi_2|_{B_1} \equiv 1$. Then the function $\varphi = \max(\varphi_1, \varphi_2)$ is continuous on $\widetilde{X}$ and separates the point x_0 and the set B . Finally, if $x_0 \in X$ and $\widetilde{f}x_0 \in \widetilde{f}B$, then the assertion follows at once from the complete regularity of X .

3) Let $\{U_n\}$ be a countable base of X and $\{V_m\}$ a countable base of Y . Then a countable base of $\widetilde{X}$ is formed by $\{U_n\}$ and all sets of the form

$$U'_{m,k} = \widetilde{f}^{-1}V_m \setminus \bigcup_{i=1}^k [U_i].$$

Corollary. Let $f : X \rightarrow Y$ be a locally perfect mapping of a Hausdorff space X onto a regular space Y . Then X is regular.

However, there exist examples of a locally perfect mapping of a completely regular, but not normal, space onto a normal space.

Theorem 4. If X and Y are metric spaces with countable bases and the mapping $f : X \rightarrow Y$ is locally perfect, then the space $\tilde{X}$ is metrizable.

Without the assumption of a countable base, the theorem is false.

Example. The space X is the interval $[0, 1]$ with the discrete topology, Y is the same interval with the usual topology, and f is the identity mapping of X onto Y . Then f is locally perfect, the space X is metric, and Y is even compact. The extended space $\tilde{X}$ is bicomact, but does not have a countable base, and therefore $\tilde{X}$ is not metrizable.

Under locally perfect mappings, metrizability, generally speaking, is preserved neither under passage to the image nor under passage to the preimage.

Theorem 5. Let X and Y be metric spaces with countable bases, with Y an absolute G_δ , and let the mapping $f : X \rightarrow Y$ be locally perfect. Then X is also an absolute G_δ .

Indeed, the extended space $\tilde{X}$ is an absolute G_δ ((1), Theorem 4), and since X is open in $\tilde{X}$, X is also an absolute G_δ .

Theorem 6. Let X and Y be finite-dimensional metric spaces with countable bases and let the mapping $f : X \rightarrow Y$ be locally perfect. If $\dim X - \dim Y = k$, then there exists a point of the space Y whose full preimage has dimension $\geq k$.

Let $\tilde{f} : \tilde{X} \rightarrow Y$ be a perfect extension of f . Since $\dim X \leq \dim \tilde{X}$, there exists a point $y_0 \in Y$ such that $\dim \tilde{f}^{-1}y_0 \geq k$. Then also $\dim f^{-1}y_0 \geq k$.

In what follows, the spaces under consideration will be assumed Hausdorff.

Let f be a mapping of X onto Y . If $X_1 \supset X$, X is dense in X_1 , and f_1 is a perfect mapping of X_1 onto Y which coincides with f on X , then f_1 will be called a perfect extension of the mapping f .

Theorem 7. Let $f : X \rightarrow Y$ be a locally perfect mapping and let $\tilde{f}_1 : \tilde{X}_1 \rightarrow Y$ be any perfect extension of it, with $\tilde{X}_1$ Hausdorff. Then: 1) X is open in $\tilde{X}_1$; 2) if $y \in Y \setminus Y^*$, then $\tilde{f}_1^{-1}y = f^{-1}y$, where Y^* is the set of points $y \in Y$ at which f is not perfect.

The proof is based on the following lemma.

Lemma. Let $\varphi : X \rightarrow Y$ be a perfect mapping. If $A \subset X$, $\varphi|A$ is perfect, and φA is closed in Y , then A is closed in X .

This follows easily from (2), Lemma 1.4.

Proof of Theorem 7. 1) Since f is locally perfect, each point x has in X a neighborhood U such that $[U]_X$ is closed in $\tilde{X}_1$ (by the lemma). There exists a

neighborhood V_x in $\tilde{X}_1$ such that $U = X \cap V$. Then $(V \setminus [U]_X) \cap X = \Lambda$, and since X is dense in $\tilde{X}_1$, $V \setminus [U]_X$ is empty. Hence $V \subset [U]_X$, i.e. $V \subset X$, $U = V$.

- 2) Let $x_0 \in \tilde{f}_1^{-1}y \setminus f^{-1}y$, $y \in Y \setminus Y^*$. Since $y \in Y \setminus Y^*$, there exists a neighborhood $U[f^{-1}y]$ such that $[U]_X$ is closed in $\tilde{X}_1$, and a neighborhood V_y such that $f^{-1}V \subseteq U$. $(\tilde{f}_1^{-1}V \setminus f^{-1}V) \cap X = \Lambda$, then $W = \tilde{f}_1^{-1}V \setminus [U]_X$ is nonempty, open in $\tilde{X}_1$, and $W \cap X = \Lambda$, which is impossible.

If the mapping $f : X \rightarrow Y$ is locally perfect, then the perfect extension $\tilde{f}$ and the space $\tilde{X}$ constructed in Theorem 1 are minimal in the following natural sense.

Theorem 8. Let $\tilde{f}_1 : \tilde{X}_1 \rightarrow Y$ be any perfect extension of the locally perfect mapping $f : X \rightarrow Y$. Then there exists a continuous mapping $\tilde{h} : \tilde{X}_1 \rightarrow \tilde{X}$ such that $\tilde{h}|_X = \text{id}$, if $x \in X$, and $\tilde{f}_1 = \tilde{f}\tilde{h}$.

Define $\tilde{h}$ as follows. If $x \in X$, then $\tilde{h}|_X = \text{id}$. If, however, $x \in \tilde{X}_1 \setminus X$, then, by Theorem 7, there exists a unique point $x' \in \tilde{X} \setminus X$ such that $\tilde{f}_1 x = \tilde{f} x'$, and then we set $\tilde{h}|_x = x' \in \tilde{X} \setminus X$. It is easy to see that $\tilde{h}$ is continuous on $\tilde{X}_1$.

Now we restrict ourselves to considering completely regular spaces. Every continuous mapping $f : X \rightarrow Y$ extends uniquely to a continuous mapping $f_B : \beta X \rightarrow \beta Y$, where βY is any bicomact extension of the space Y . The mapping f_B is perfect on the set $f_B^{-1}Y$. Denote $f_B^{-1}Y$ by $\tilde{X}_0$; $\tilde{X}_0$ is completely regular. Let $f_B|_{\tilde{X}_0} = \tilde{f}_0$. The space X is contained in $\tilde{X}_0$ and is dense in it; hence the mapping $\tilde{f}_0 : \tilde{X}_0 \rightarrow Y$ is a perfect extension of the mapping $f : X \rightarrow Y$. The extension $\tilde{f}_0$ and the space $\tilde{X}_0$ are maximal, namely:

Theorem 9. Let $\tilde{f}_1 : \tilde{X}_1 \rightarrow Y$ be any perfect extension of the mapping $f : X \rightarrow Y$, where X , Y , and $\tilde{X}_1$ are completely regular. Then there exists a continuous mapping $\tilde{h} : \tilde{X}_0 \rightarrow \tilde{X}_1$ such that $\tilde{h}|_X = \text{id}$, if $x \in X$, and $\tilde{f}_0 = \tilde{f}_1 \tilde{h}$.

We shall call a mapping $f : X \rightarrow Y$ **locally closed** if every point $x \in X$ has a neighborhood U such that $f[U]$ is closed in Y , the mapping $f|_{[U]}$ is closed, and $[U] \cap \text{Fr } f^{-1}y$ is a bicomactum for any point $y \in f[U]$.

Theorem 10. Let f be a locally closed mapping of a metric space X onto a space Y satisfying the first axiom of countability. Then there exist a space $\tilde{X} \supset X$ and a closed mapping $\tilde{f} : \tilde{X} \rightarrow Y$ such that: 1) $\tilde{f}|_X = f$; 2) $\tilde{f}(\tilde{X} \setminus X) = Y^*$, where Y^* is the set of points $y \in Y$ at which f is not closed; 3) $\tilde{f}|_{(\tilde{X} \setminus X)}$ is one-to-one; 4) X is dense in $\tilde{X}$; 5) X is open in $\tilde{X}$. The conditions 1, 2, and 3 determine $\tilde{X}$ and $\tilde{f}$ uniquely.

The space $\tilde{X}$ is obtained from X as follows: to each full inverse image of a point $y \in Y^*$ we add one new point, and to the full inverse images of points $y \in Y \setminus Y^*$ we add nothing. We leave the topology on X unchanged. If $x \in \tilde{X} \setminus X$, its neighborhoods will be the sets

of the form $U = \tilde{f}^{-1}V \setminus F$, where $V = V_y$ is an arbitrary neighborhood of the point $y = \tilde{f}x$ in Y , $F \subset X$ and is closed in X , fF is closed in Y , the mapping $f|F$ is closed, and $F \cap \text{Fr } f^{-1}y$ is bicomact for $y \in fF$.

Analogously to Theorem 1, the closedness of $\tilde{f}$ is proved.

If the space Y is Hausdorff (respectively regular), then $\tilde{X}$ is also Hausdorff (respectively regular).

Theorem 11. *Let f be a locally closed mapping of a normal space X onto a space Y with the first axiom of countability. Then the boundary of the full preimage $f^{-1}y$ of each point y is locally compact.*

If f is a locally perfect mapping of a metrizable space X onto a space Y with the first axiom of countability, then, extending it by the method described in Theorem 10, we obtain a closed but, generally speaking, not perfect extension.

I take this opportunity to express my deep gratitude to my adviser I. A. Vainshtein for constant assistance and valuable advice, which had a great influence on the direction of this work.

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Received
8 VII 1966

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Note: Figure translations are in progress. See original paper for figures.

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