

ON EXACT SOLUTIONS OF EQUATIONS DESCRIBING THE MOTIONS OF A GRAVITATING GAS WITH HOMOGENEOUS DEFORMATION AND A TRANSVERSAL VELOCITY COMPONENT

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Abstract

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AERODYNAMICS

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ON EXACT SOLUTIONS OF EQUATIONS DESCRIBING THE MOTIONS OF A GRAVITATING GAS WITH HOMOGENEOUS DEFORMATION AND A TRANSVERSAL VELOCITY COMPONENT

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Nonstationary spherically symmetric gas flows with homogeneous deformation, i.e., flows in which the radial velocity of a gas particle is proportional to its distance from the center of symmetry, were first considered by L. I. Sedov ^(1,2). He obtained a class of particular exact solutions of the equations describing such motions and gave a classification and detailed study of these motions. Analogous solutions for the case in which the forces of Newtonian attraction between gas particles are taken into account were found by M. L. Lidov ⁽³⁾. Yu. P. Ladikov ⁽⁴⁾ extended the class of these solutions by considering the rotation of gas particles about an axis passing through the center of symmetry ($v_\varphi \neq 0$). In the present note new solutions of an analogous type are presented.

The equations of gas dynamics in spherical coordinates, taking account of axial symmetry ($\partial/\partial\varphi = 0$) and $v_\theta = 0$, have the form

$$\begin{aligned} \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} - \frac{v_\varphi^2}{r} + \frac{1}{\rho} \frac{\partial p}{\partial r} - \frac{\partial \psi}{\partial r} &= 0, \\ \frac{\partial v_\varphi^2}{\partial t} + v_r \frac{\partial v_\varphi^2}{\partial r} + 2v_r \frac{v_\varphi^2}{r} &= 0, \\ v_\varphi^2 \operatorname{ctg} \theta - \frac{1}{\rho} \frac{\partial p}{\partial \theta} + \frac{\partial \psi}{\partial \theta} &= 0, \\ \frac{\partial \rho}{\partial t} + v_r \frac{\partial \rho}{\partial r} + \rho \left(\frac{\partial v_r}{\partial r} + \frac{2v_r}{r} \right) &= 0, \end{aligned} \tag{1}$$

$$\frac{\partial p}{\partial t} + v_r \frac{\partial p}{\partial r} - \frac{\gamma p}{\rho} \left(\frac{\partial \rho}{\partial t} + v_r \frac{\partial \rho}{\partial r} \right) = 0,$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) = -4\pi G \rho.$$

Here v_r and v_φ are the radial and transversal velocity components; p is pressure; ρ is density; $\gamma = c_p/c_v$ is the ratio of specific heats; G is the gravitational constant; ψ is the gravitational potential.

Assuming, as in ⁽¹⁻⁴⁾, that the radial velocity v_r depends linearly on the coordinate r , we set

$$v_r = \frac{r}{\mu} \frac{d\mu}{dt}, \quad (2)$$

where $\mu(t)$ is a certain function of time, to be determined below. Each particle with Lagrangian coordinate ξ in such a motion will undergo homogeneous deformation

$$r = \xi \mu(t). \quad (3)$$

If $\gamma = 5/3$ and $\partial\psi/\partial\theta = 0$, then equations (1) admit the particular solution

$$v_r = \xi \frac{d\mu}{dt}, \quad v_\varphi^2 = \frac{\xi \sin \theta}{\mu^2} \frac{d\Phi}{dx}, \quad \rho = \frac{\rho_0}{\mu^3}, \quad (4)$$

$$p = \frac{\rho_0}{\mu^5} \left[\Phi(x) + b_1 \frac{\xi^2}{2} + b_2 \right].$$

Here $\Phi(x)$ is an arbitrary function, $x = \xi \sin \theta$ is the distance of the particle from the axis of symmetry. The dependence of μ on t is determined from the differential equation

$$\left(\frac{d\mu}{dt} \right)^2 = \frac{b_1}{\mu^2} + \frac{b_3}{\mu} + b_4 = f(\mu), \quad (4')$$

where $b_1, \dots, b_4$ are constants ($b_3 > 0$), and the constant ρ_0 is related to b_3 by the formula $\rho_0 = \frac{3}{8\pi G} b_3$.

Depending on the values of the constants b_1, b_3, b_4 , the solution (4)–(4') may describe motions of various types, a detailed analysis of which is given in papers ⁽⁵⁻⁷⁾.

In particular, if $b_1 < 0$ and $b_3^2 > 4b_1b_4$, the function $f(\mu)$ may have two distinct roots $\mu_1 > 0$ and $\mu_2 > \mu_1$. The solution (4)–(4') in this case describes gas motion of a pulsating type: gas particles rotating with variable velocity v_φ about the axis of symmetry approach this axis to the distance $r_1 = \xi\mu_1$, and then recede from it to the distance $r_2 = \xi\mu_2$. Since this motion is periodic (the period depends on the constants b_1, b_3, b_4), it may be of interest in connection with the study of pulsations of variable stars (see ⁽²⁾, p. 325).

We note that, because of the presence in (4) of the arbitrary function $\Phi(x)$, which characterizes the initial distribution of the transverse component of the velocity, the solution (4)–(4') is more general than the one published in paper ⁽⁴⁾.

In those cases when gravitational forces may be neglected, equations (1), if one puts $G = 0$ in them, have a solution containing two arbitrary functions $\Phi_1(x)$ and $\Phi_2(\xi)$, which characterize the initial distributions of density and of the transverse component of the velocity. This solution has the form (the prime denotes differentiation with respect to ξ)

$$\begin{aligned} v_r &= \xi \frac{d\mu}{dt}, & v_\varphi^2 &= \frac{\sin \theta}{\mu^2} \frac{\xi^2}{\Phi_2'(\xi)} \frac{d\Phi_1}{dx}, & \rho &= \frac{\Phi_2'(\xi)}{\xi\mu^3}, \\ p &= \frac{\Phi_1(x) + a_1\Phi_2(\xi) + a_2}{\mu^5} & \left(\xi = \frac{r}{\mu}, \quad x = \xi \sin \theta \right). \end{aligned} \quad (5)$$

The dependence of μ on t is determined by the equation

$$\left(\frac{d\mu}{dt} \right)^2 = \frac{a_1}{\mu^2} + a_3. \quad (5')$$

Here a_1, a_2, a_3 are arbitrary constants.

If, instead of the adiabaticity condition (the fifth equation of system (1)), one uses the condition $\partial T / \partial r = 0$, i.e., instead of adiabatic motions of the gas one considers homothermic motions, then it is not difficult to show that system (1) has no solutions analogous to the solutions (4)–(5).

It is also of interest to obtain solutions of the equilibrium equations

$$\begin{aligned} \frac{v_{\varphi 1}^2}{r} - \frac{1}{\rho_1} \frac{\partial p_1}{\partial r} + \frac{\partial \psi}{\partial r} &= 0, \\ v_{\varphi 1}^2 \operatorname{ctg} \theta - \frac{1}{\rho_1} \frac{\partial p_1}{\partial \theta} + \frac{\partial \psi}{\partial \theta} &= 0, \\ \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) &= -4\pi G \rho_1, \end{aligned} \quad (6)$$

which follow from system (1) when $v_r = 0$ and $\partial/\partial t = 0$. Let $p_1 = p_1(r, \theta)$, $\rho_1 = \rho_1(r, \theta)$. Then, using the first two equations of (6), we obtain equation

$$\frac{1}{\operatorname{ctg} \theta} \left(\frac{1}{\rho_1} \frac{\partial p_1}{\partial \theta} - \frac{\partial \psi}{\partial \theta} \right) - r \left(\frac{1}{\rho_1} \frac{\partial p_1}{\partial r} - \frac{\partial \psi}{\partial r} \right) = 0, \quad (7)$$

which can be integrated if one assumes that $\rho_1 = \text{const} = \rho_0$. The integral of equation (7) in this case has the form

$$p_1/\rho_0 - \psi = F(z), \quad (8)$$

where $F(z)$ is an arbitrary function, $z = r \sin \theta$.

If now the third equation of system (6) is differentiated with respect to θ , assuming that $\rho_1 = \rho_1(r)$, and $\partial\psi/\partial\theta = R(r)\Pi(\theta)$, we obtain the equation

$$2r \frac{R'}{R} + r^2 \frac{R''}{R} = \frac{1}{\sin^2 \theta} - \frac{\dot{\Pi}}{\Pi} \operatorname{ctg} \theta - \frac{\ddot{\Pi}}{\Pi} = a = \text{const} \quad (9)$$

(here a prime denotes differentiation with respect to r , and a dot differentiation with respect to θ). Solving (9), we find

$$\begin{aligned} \partial\psi/\partial r &= (2c_2/r^3 - c_1) \cos \theta - \chi'(r), \\ \partial\psi/\partial \theta &= (c_1 r + c_2/r^2) \sin \theta, \\ \psi(r, \theta) &= -(c_1 r + c_2/r^2) \cos \theta - \chi(r), \end{aligned} \quad (10)$$

where c_1 and c_2 are arbitrary constants, and the function $\chi(r)$ is related to the function $\rho_1(r)$ by the formula $(r^2 \chi')' = 4\pi G r^2 \rho_1$. Substituting $\partial\psi/\partial r$ and $\partial\psi/\partial \theta$ from (10) into (7), we obtain

$$r \frac{\partial p_1}{\partial r} - \frac{1}{\operatorname{ctg} \theta} \frac{\partial p_1}{\partial \theta} = \varphi(r, \theta), \quad (11)$$

where

$$\varphi(r, \theta) = \left\{ -\frac{c_1 r}{\cos \theta} + \frac{c_2}{r^2} \frac{3 \cos^2 \theta - 1}{\cos \theta} - r \chi'(r) \right\} \rho_1(r).$$

By the change of variables $\ln r = \xi$, $\ln \sin \theta = \eta$, equation (11) is brought to the form

$$\frac{\partial p_1}{\partial \xi} - \frac{\partial p_1}{\partial \eta} = \bar{\varphi}(\xi, \eta). \quad (12)$$

If φ does not depend on θ , i.e. $c_1 = c_2 = 0$ (and consequently $\partial\psi/\partial\theta = 0$), then from equation (12) one can find $p_1(\xi, \eta)$, and hence the equilibrium equations (6) in this case have the solution

$$v_{\varphi 1}^2 = \frac{z}{\rho_1(r)} \frac{d\chi(z)}{dz}, \quad p_1 = \chi(z) - \int \rho_1(r) \chi'(r) dr, \quad (13)$$

containing two arbitrary functions $\rho_1(r)$ and $\chi(z)$, where $z = r \sin \theta$.

If, however, $\rho_1 = \text{const} = \rho_0$, then from (8), (10), and (6) we find the following solution of equations (6), containing one arbitrary function $F(z)$:

$$v_{\varphi 1}^2 = z \frac{dF(z)}{dz}, \quad p_1 = \rho_0 \left[F(z) - \left(c_1 r + \frac{c_2}{r^2} \right) \cos \theta - \frac{2\pi G}{3} \rho_0 r^2 \right]. \quad (14)$$

The exact solutions (13)–(14) of the equilibrium equations (6) may be used to prescribe initial distributions of the characteristics of a gravitating gas consisting of particles rotating about an axis passing through the center of symmetry.

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