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L. A. Nazarova, A. V. Roiter

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Abstract

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MATHEMATICS

L. A. NAZAROVA, A. V. ROITER

A REFINEMENT OF A THEOREM OF BASS

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Let A be a local (commutative) Noetherian ring with identity satisfying the following conditions: a) the Krull dimension of the ring A is not greater than 1; b) A has no nilpotent elements; c) the integral closure of the ring A is a finitely generated A -module.

In § 7 of Bass' s paper ⁽¹⁾, conditions are considered under which every indecomposable finitely generated torsion-free A -module is isomorphic to an ideal of the ring A . This question is of interest, in particular, from the point of view of the theory of integral representations. The methods developed in ⁽¹⁾ are very interesting; however, there is an inaccuracy in the formulation of the result, and correcting it is the aim of the present paper.

From the restrictions imposed on the ring A it follows that the ring is a subring of a direct sum of local Dedekind rings

$$A \subset \bigoplus_{i=1}^l A_i.$$

We shall say that the ring A **satisfies condition α** (or, for brevity, is an **α -ring**) if every finitely generated, indecomposable, torsion-free (torsionless) A -module (i.e., a module that can be embedded in a free module) is isomorphic to an ideal of the ring A . In what follows, speaking of modules, we shall have in mind finitely generated torsion-free modules.

It follows from Day' s work ⁽²⁾ that for every α -ring $l \leq 3$. In ⁽¹⁾ it is proved that for $l = 1, 2$ the ring A satisfies condition α if and only if every ideal of the ring A has at most two generators. Bass further asserts that for $l = 3$ the ring A satisfies condition α if and only if A is a triad of Dedekind rings. By a triad T of Dedekind rings A_1, A_2, A_3 with isomorphic residue field k and given homomorphisms $\varepsilon_i : A_i$ onto k , Bass calls the set of triples (a_1, a_2, a_3) , $a_i \in A_i$, for which $a_1\varepsilon_1 = a_2\varepsilon_2 = a_3\varepsilon_3$. However, there exist (for $l = 3$) α -rings that are not triads. For example, the integral group ring of the cyclic group of order 4 is an α -ring ⁽³⁾ and is not a triad. We shall give a certain description of all α -rings (for $l = 3$) that are not triads.

Thus, let A be an α -ring for which $l = 3$, $A \subset A_1 \oplus A_2 \oplus A_3$. Denote, following ⁽¹⁾, $\mathfrak{P}_i = A \cap (A_j \oplus A_k)$; $D_i = A \cap A_i = \mathfrak{P}_j \cap \mathfrak{P}_k$; $\mathfrak{M}$ is the maximal ideal of the ring A ; π_i is a prime element of A_i ; $k \simeq A/\mathfrak{M} \simeq A_i/\pi_i A_i$ is the residue field, $i \neq j \neq k$; $i, j, k = 1, 2, 3$.

In ⁽¹⁾ it is proved (and we shall use this) that if A is an α -ring ($l = 3$), then:

- 1) $A/\mathfrak{P}_i = A_i$;
- 2) $\mathfrak{P}_1 + \mathfrak{P}_2 = \mathfrak{P}_1 + \mathfrak{P}_3 = \mathfrak{P}_2 + \mathfrak{P}_3 = \mathfrak{M}$.

From 1) it follows, in particular, that D_i (as an A -module) is isomorphic to A_i , and also that A is a subring of some triad T .

Using 1), 2), it is not difficult to show that

$$\text{Ext}(A_i, A_j) \simeq k.$$

Denote $\overline{\mathfrak{P}}_j = A/A \cap A_j$, $j = 1, 2, 3$. Obviously, $\text{Ext}(\overline{\mathfrak{P}}_j, A_j)$ either is cyclic and (since $\overline{\mathfrak{P}}_j$ is an extension of A_i by A_k) is isomorphic to k , or is an extension of k by k . In the latter case

$$\text{Ext}(\overline{\mathfrak{P}}_j, A_j) \cong A_j/\pi_j^2 A_j, \quad j = 1, 2, 3.$$

Let X be an arbitrary A -module. Put $X_i = \{x, xD_i = 0\}$, $\overline{X}_i = X/X_i$. $\overline{X}_i$ is an A_i -module, since $\overline{X}_i \mathfrak{P}_i = 0$. X_i is a $\mathfrak{P}_i$ -module, since $X_i D_i = 0$. It is not difficult to verify that $\overline{X}_i$ is a finitely generated torsion-free A_i -module, and X_i is a finitely generated torsion-free $\mathfrak{P}_i$ -module. Since every finitely generated torsion-free module over a local Dedekind ring is isomorphic to a direct sum of rings, we have $\overline{X}_i = A_i^{(n)}$ ($A_i^{(n)}$ is the direct sum of n copies of A_i). Using the fact that $\text{Ext}(A_j, A_k) \cong k$, it is not difficult to show that

$$X_i + \overline{\mathfrak{P}}_i^{(m_1)} \oplus A_j^{(m_2)} \oplus A_k^{(m_3)}, \quad i \neq j \neq k.$$

Consequently,

$$\text{Ext}(\overline{X}_i, X_i) = \text{Ext}(A_i, \overline{\mathfrak{P}}_i)^{(nm_1)} \oplus \text{Ext}(A_i, A_j)^{(nm_2)} \oplus \text{Ext}(A_i, A_k)^{(nm_3)}.$$

Using the technique customary in the theory of integral representations (see, for example, (4)), one can show that if $\text{Ext}(A_i, \overline{\mathfrak{P}}_i) \cong k$, then A has 8 indecomposable modules; if, however, $\text{Ext}(A_i, \overline{\mathfrak{P}}_i) \cong A_j/\pi_j^2 A_j$, then A has 9 indecomposable modules. In both cases all these modules are isomorphic to ideals of the ring A . $A \subseteq T$, and hence every T -module is an A -module. The triad has 8 indecomposable modules (1). If $A \subset T$, then A must have at least 9 indecomposable

modules (8 T -modules and A itself). Thus, if A has 8 indecomposable modules, then A is a triad.

Suppose now that A is not a triad and, consequently, has 9 indecomposable modules. $A \subset T$. We shall show that A is a maximal submodule of the triad T , considered as an A -module. A is a ring with identity, and hence it cannot be contained in any T -submodule of the triad T . Since every A -module is a direct sum of a T -module and of the module $A^{(n)}$, every module lying between A and T would have to be isomorphic to A . In this case the ring A would be a maximal submodule of some module isomorphic to A . Since the module A has one maximal submodule, it would follow that A is a principal-ideal ring, which of course cannot be the case, if only because $l = 3$. From the same considerations it follows that the maximal ideal $\mathfrak{M}$ of the ring A is a T -submodule. It is not difficult to show that if $\mathfrak{M}$ were decomposable, then A would be a triad. Consequently, $\mathfrak{M}$ is a maximal indecomposable T -ideal in the maximal ideal $\mathfrak{M}(T)$ of the triad T .

Simple computations show that in $\mathfrak{M}$ one can choose three generators: $(\pi_1^2, 0, 0)$; $(\pi_1 X_2, \pi_2, 0)$; $(\pi_1 X_3, 0, \pi_3)$, where X_2, X_3 are elements of the ring A_1 not lying in the prime ideal. In other words,

$$\mathfrak{M} = \{(\pi_1 a_1, \pi_2 a_2, \pi_3 a_3), \bar{a}_1 = \bar{a}_2 X_2 + \bar{a}_3 X_3\},$$

where $\bar{a}_1, \bar{a}_2, \bar{a}_3$ are elements of the residue field k to which the elements a_1, a_2, a_3 pass under the ring homomorphisms A_1, A_2, A_3 onto k fixed in the definition of the triad; X_2, X_3 are certain nonzero elements of k .

Thus, the ring A must lie between T and $\mathfrak{M}$, with T/A isomorphic to $A/\mathfrak{M}$ as an A -module. From the form of $\mathfrak{M}$ we conclude that if such a ring A exists, then it satisfies conditions 1), 2) and, consequently, as we have shown, all its indecomposable modules are isomorphic to ideals. Therefore, in order to describe all α -rings (for $l = 3$) that are not triads, it remains to check for which A_1, A_2, A_3, X_2, X_3 there exists a ring A such that $T/A \cong A/\mathfrak{M}$, in other words, in what case the factor ring $\bar{T} = T/\mathfrak{M}$ contains a subfield $\bar{A}$ such that $(\bar{T} : \bar{A}) = 2$. Note that $\bar{T}$ has one ideal $\mathfrak{M}(T)$.

Lemma. Let Λ be a commutative ring with identity with a unique nontrivial ideal $I \neq p\Lambda$. Then Λ contains a subfield $\bar{k}$ such that

$$(\Lambda : \bar{k}) = 2.$$

Let the characteristic of $k = \Lambda/I$ be p . It is not difficult to show that the set of p -th powers forms a subfield k' of the ring Λ . Using Zorn's lemma, choose a maximal field $\bar{k}$ among the subfields of the ring Λ containing k' . Let φ be the natural mapping of Λ onto k . Suppose that $\varphi(\bar{k}) \neq k$. Then there is a simple extension of the field $\varphi(\bar{k})$ contained in k . Considering separately the cases where this extension is transcendental, algebraic separable, and purely inseparable, one can show in each case that $\bar{k}$ is not maximal. In the last case

one must use the fact that $\bar{k} \supset k'$. Hence $\varphi(\bar{k}) = k$, $(\Lambda : \bar{k}) = 2$. In the case where the characteristic of k is 0, the proof is simplified because of the absence of inseparable extensions.

We also note that in the case where k is a finite field, $k = k'$, and therefore $\bar{k}$ is determined uniquely.

Thus, if the characteristic of k is 0, there always exists A such that

$$T/A \cong A/\mathfrak{M}.$$

If, however, the characteristic of k is p , then for the existence of such a ring it is necessary and sufficient that

$$\mathfrak{M}(T) \neq pT,$$

i.e. that

$$(p, p, p) \in \mathfrak{M}.$$

Let the characteristic of k be p . Consider the ideal pA_i in A_i . If the characteristic of A_i is 0, then $pA_i \neq 0$ and there is a number s_i such that

$$pA_i = \pi_i^{s_i} A_i.$$

If the characteristic of A_i is p , then we agree to regard s_i as equal to ∞ . In the case $s_i = 1$ we shall regard $\pi_i = p$. We agree that

$$s_1 \geq s_2 \geq s_3.$$

Simple calculations show that, for $s_i \geq 2$ ($i = 1, 2, 3$), a ring A with the properties we require exists for arbitrary $\bar{X}_2, \bar{X}_3$; for $s_1 \geq 2, s_2 \geq 2, s_3 = 1$ no such ring exists for any $\bar{X}_2, \bar{X}_3$; for $s_1 \geq 2, s_2 = s_3 = 1$ the ring A exists when

$$\bar{X}_2 + \bar{X}_3 = 0;$$

for

$$s_1 = s_2 = s_3 = 1$$

the ring exists when

$$\bar{X}_2 + \bar{X}_3 = 1.$$

In the last case, for $p = 2$, such a ring obviously does not exist ($\bar{X}_2, \bar{X}_3$ are distinct from zero) for any $\bar{X}_2, \bar{X}_3$; when the characteristic is different from 2, the corresponding $\bar{X}_2, \bar{X}_3$ can, of course, be chosen.

Thus, if A_1, A_2, A_3 are three local Dedekind rings with isomorphic residue field, then, except for the case where one of the ramification coefficients is equal to 1 and the others are not equal to 1, and the case where all ramification coefficients are equal to 1 and the characteristic of the residue field is 2, in the direct sum

$$A_1 \oplus A_2 \oplus A_3$$

there are α -rings distinct from triads. Every such ring can be obtained in the following way: take a certain triad T of the given rings A_1, A_2, A_3 and construct in it a submodule

$$\mathfrak{M} = \{(\pi_1 a_1, \pi_2 a_2, \pi_3 a_3), \bar{a}_1 = \bar{a}_2 \bar{X}_2 + \bar{a}_3 \bar{X}_3\},$$

where $\bar{X}_2, \bar{X}_3$ are some nonzero elements of the residue field, with

$$\bar{X}_2 + \bar{X}_3 = 1$$

when $s_1 = s_2 = s_3$, and

$$\bar{X}_2 + \bar{X}_3 = 0$$

when $s_1 \geq 2, s_2 = s_3 = 1$.

If k is a finite field, then

$$A = T^p + \mathfrak{M}.$$

If k is an arbitrary field, then choose in the factor ring

$$\bar{T} = T/\mathfrak{M}$$

some subfield $\bar{k}$ such that

$$(\bar{T} : \bar{k}) = 2.$$

Then A is the full preimage of $\bar{k}$ under the natural homomorphism

$$T \rightarrow \bar{T}.$$

Institute of Mathematics
Academy of Sciences of the Ukrainian SSR

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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