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ON STRICTLY COSINGULAR OPERATORS

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Abstract

Full Text

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MATHEMATICS

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ON STRICTLY COSINGULAR OPERATORS

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All spaces under consideration are assumed to be Banach spaces, and all operators linear and bounded. An operator $A : X \rightarrow Y$ is called a $\Phi_+(\Phi_-)$ -operator if $\text{Im } A$ is closed and $\dim \text{Ker } A < \infty$ ($\text{codim Im } A < \infty$). An operator $T : X \rightarrow Y$ is called **strictly singular** ⁽²⁾ if there does not exist an infinite-dimensional Banach space E with isomorphic embeddings $i_1 : E \rightarrow X$, $i_2 : E \rightarrow Y$ such that $T i_1 = i_2$. An operator $T : X \rightarrow Y$ is called **strictly cosingular** ⁽³⁾ if there does not exist an infinite-dimensional Banach space E with quotient maps $h_1 : X \rightarrow E$ and $h_2 : Y \rightarrow E$ such that $h_2 T = h_1$. An operator $B : X \rightarrow Y$ is called a $\Phi_+(\Phi_-)$ -**admissible perturbation** if $A + B$ is a $\Phi_+(\Phi_-)$ -operator for every $\Phi_+(\Phi_-)$ -operator $A : X \rightarrow Y$. It is known (see ^(1,2)) that all strictly singular operators are Φ_+ -admissible perturbations. In the present note it is established that all strictly cosingular operators are Φ_- -admissible perturbations.

Following V. Pták, we shall use the notation $X \subseteq Y$ to express that X is a closed subspace of Y . The space of all bounded linear operators acting from X to Y will be denoted by $L(X, Y)$.

Lemma 1. Let $T \in L(X, Y)$. Then:

- a) T is strictly singular $\iff$ there does not exist an infinite-dimensional Banach space E with quotient maps $h_1 : X^* \rightarrow E$ and $h_2 : Y^* \rightarrow E$ such that $\text{Ker } h_1$ is weakly closed and $h_1 T^* = h_2$.
- b) T is strictly cosingular $\iff$ the restriction of T^* to any infinite-dimensional weakly closed subspace of Y^* is not an isomorphic embedding.

Lemma 2. Let $T \in L(X, Y)$. If the restriction of T^* to a weakly closed $Z_1 \subseteq Y^*$ is not a Φ_+ -operator, then for every $\varepsilon > 0$ there exists an infinite-dimensional weakly closed $Z_2 \subseteq Z_1$ such that the restriction of T^* to Z_2 is completely continuous and has norm less than ε .

The proof is carried out analogously to the proof of Theorem 4.1 in ⁽¹⁾. Biorthogonal sequences $\{y_k^*\}_1^\infty$ ($y_k^* \in Z_1$) and $\{y_k\}_1^\infty$ ($y_k \in Y$) are constructed such that

$$|y_k^*| = 1, \quad |y_k| < 2^{2k-1}, \quad |T^* y_k^*| < \varepsilon \cdot 2^{1-3k} \quad (k = 1, \dots, n, \dots).$$

Define the operator $A : Y^* \rightarrow X^*$ by the equality

$$Ay^* = \sum_{k=1}^{\infty} \langle y_k, y^* \rangle T^* y_k^*.$$

It is clear that A is completely continuous and $|A| < \varepsilon$. But $A = B^*$, where

$$Bx = \sum_{k=1}^{\infty} \langle x, T^* y_k^* \rangle y_k.$$

Therefore $\text{Ker}(T^* - A) = \text{Ker}(T - B)^*$ is weakly closed, and

$$Z_2 = Z_1 \cap \text{Ker}(T^* - A)$$

is weakly closed. Since all $y_k^* \in Z_2$, it follows that $\dim Z_2 = \infty$. The restriction of T^* to Z_2 is completely continuous and has norm less than ε .

Theorem 1. Let $T \in L(X, Y)$. The following conditions are equivalent:

- a) T is a strictly cosingular operator;
- b) for every infinite-dimensional weakly closed $Z \subseteq Y^*$, the restriction of T^* to Z is not a Φ_+ -operator;
- c) for every infinite-dimensional weakly closed $Z_1 \subseteq Y^*$ there exists an infinite-dimensional weakly closed $Z_2 \subseteq Z_1$ such that the restriction of T^* to Z_2 is completely continuous.

Corollary 1. All strictly cosingular operators are Φ_- -admissible perturbations. The set of all strictly cosingular operators from $L(X, Y)$ forms a closed subspace in $L(X, Y)$, which is a two-sided ideal if $X = Y$.

Proof. This follows from Proposition 1 in ⁽³⁾, Theorem 1, and Lemma 2.

From this corollary and Theorem 5.1 in ⁽¹⁾ it follows that

Corollary 2. In the spaces l_p ($p \geq 1$) and c_0 , strictly cosingular operators are completely continuous.

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