

ALMOST-PERIODICITY OF BOUNDED SOLUTIONS OF NONLINEAR SYSTEMS

V. M. Cheresiz

1967

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196701.68742>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.925

MATHEMATICS

V. M. CHERESIZ

ALMOST-PERIODICITY OF BOUNDED SOLUTIONS OF NONLINEAR SYSTEMS

(Presented by Academician L. S. Pontryagin on 2 III 1966)

The present note is a continuation of the author's work ⁽⁸⁾. Let the system be given

$$\dot{x} = F(x, t), \quad (1)$$

where $F(x, t)$ satisfies the local existence and uniqueness theorems in the cylinder $C_\Gamma\{x \in \Gamma, t \in I\}$; here $\Gamma \subset R^n$ is a domain of n -dimensional Euclidean space, and I is the time axis $(-\infty < t < +\infty)$. Suppose there is a scalar function $V(t, x, y) \in C^{(1)}(I \times \Gamma \times \Gamma)$, taking values, generally speaking, of different signs, whose set of zeros for every t contains the "diagonal" $(x = y)$ of the space $R^{2n}(x, y)$. Extending the known Lyapunov definitions to the case of a pair of vector arguments, we shall say that $V(t, x, y)$ is positive definite in $\Gamma \times \Gamma$ if

$$V(t, x, y) \geq W(x, y), \quad (x, y) \in \Gamma \times \Gamma, t \in I, \quad (2)$$

where $W(x, y)$ is continuous and

$$W(x, y) > 0 \quad (x \neq y); \quad W(x, x) = 0. \quad (3)$$

We shall call the vector-function $F(x, t)$ V -monotone in x in Γ for all $t \in I$, if the function

$$\begin{aligned} \dot{W}(t, x, y) = \frac{\partial V}{\partial t} + \sum_{i=1}^n \frac{\partial V}{\partial x_i} f_i(x, t) + \sum_{i=1}^n \frac{\partial V}{\partial y_i} f_i(y, t) = \frac{\partial V}{\partial t} \\ + (\text{grad}_x V, F(x, t)) + (\text{grad}_y V, F(y, t)) \end{aligned} \quad (4)$$

is negative outside the "diagonal," i.e.

$$\dot{W}(t, x, y) < 0 \quad (x \neq y, t \in I); \quad \dot{W}(t, x, x) = 0 \quad (t \in I). \quad (5)$$

In this case $F(x, t)$ is uniformly V -monotone in Γ if $\dot{W}(t, x, y)$ is negative definite. Finally, we shall call system (1) V -monotone in C_Γ if $F(x, t)$ is such. We shall say that $V(t, x, y)$ admits in $\Gamma \times \Gamma$ an upper bound that is infinitesimal in a neighborhood of the “diagonal” ($x = y$), if

$$|V(t, x, y)| \leq W(x, y), \quad (x, y) \in \Gamma \times \Gamma, \quad t \in I \quad (6)$$

with a function $W(x, y)$ of type (3).

In what follows, precisely such $V(t, x, y)$ are considered everywhere.

Generalizing the definition of the A -property introduced by N. N. Krasovskii ⁽⁴⁾, we shall say that in the domain Γ the A -property is satisfied if, for each prescribed bounded domain H ($\bar{H} \subset \Gamma$) and for any natural k , one can indicate a number $T(H, k) > 0$ such that, for any pair of trajectories of system (1), $x(t, x_0, t_0)$ and $y(t, y_0, t_0)$, at least one leaves H in a time interval not exceeding $T(H, k)$, provided only that $\|x_0 - y_0\| \geq$

$\geq 1/k$. At the same time, obviously, $C_{\bar{H}}$ can contain no more than one solution $\Phi(t)$ of system (1) that is bounded on the entire axis I . In Theorems 1 and 2 we shall assume that in every bounded $\bar{H} \subset \Gamma$ the Lipschitz condition is satisfied:

$$\|F(x, t) - F(y, t)\| \leq L(\bar{H})\|x - y\|.$$

Theorem 1. In order that the A -property hold in the domain Γ , it is necessary and sufficient that in every bounded domain $\bar{H}$ ($\bar{H} \subset \Gamma$) system (1) be uniformly V -monotone with a function $V(t, x, y)$ having continuous and uniformly bounded partial derivatives in $I \times \bar{H} \times \bar{H}$:

$$\partial V / \partial x_i, \quad \partial V / \partial y_i, \quad \partial V / \partial t \quad (i = 1, \dots, n).$$

Theorem 2. If $F(x, t)$ is uniformly V -monotone and almost periodic in t uniformly in $\bar{H}_0$ (a bounded domain, $\bar{H}_0 \subset \Gamma$) with Fourier-exponent module $M_{\bar{H}_0}$, and system (1) admits in $C_{\bar{H}_0}$ a bounded solution $\Phi(t)$, then this solution is an almost-periodic vector function with module

$$M_\Phi \subseteq M_{\bar{H}_0}. \quad (7)$$

If $V(x, y)$ does not depend on t , then the assertion of the theorem is true even under simple V -monotonicity (without the requirement of its uniformity).

In the following theorem it is assumed that all systems of the class $H\{F(x, t)\}$ satisfy the local uniqueness theorem.

Theorem 3. Every uniformly asymptotically stable ⁽⁴⁾ solution $\Phi(t)$ of the almost-periodic system (1), whose trajectory is the unique trajectory wholly

contained in some bounded domain H_0 , is an almost-periodic vector function satisfying the inclusion (7).

From this theorem follow the results of A. Halanay ⁽²⁾ and T. Yoshizawa ⁽³⁾ for systems of ordinary differential equations.

We shall say that system (1) has the property of convergence if it has a solution $\Phi(t)$, bounded on the entire axis I , uniformly asymptotically stable and stable in the large, i.e., for any other solution $x(t, x_0, t_0)$ one has

$$\|X(t, x_0, t_0) - \Phi(t)\| \rightarrow 0 \quad (t \rightarrow +\infty). \quad (8)$$

Moreover, if $F(x, t)$ is almost periodic in t uniformly on any compact set in x and the limiting solution $\Phi(t)$ is also an almost-periodic vector function, then we shall say that **almost-periodic convergence** takes place. This definition differs from the known definition ⁽¹⁾ by the absence of the requirement that attraction from any cylinder $C_R\{\|x\| \leq R, t \in I\}$ be uniform; convergence ⁽¹⁾ could be called **uniform**.

Here the uniformity of attraction, by the definition of uniform asymptotic stability, is required only from some arbitrarily "narrow" δ -tube of the solution $\Phi(t)$.

Theorem 4. Every almost-periodic system with convergence is a system with almost-periodic convergence.

Let us note that T. Yoshizawa ⁽³⁾ and G. Seifert ⁽⁷⁾ require uniform convergence and uniform boundedness of solutions for the almost-periodicity of a bounded solution; Theorem 4 does not require this.

Theorem 5. Every uniformly Lyapunov-stable, asymptotically stable, bounded solution of an ω -periodic system is uniformly asymptotically stable.

In the particular case when the bounded solution is assumed in advance to be ω -periodic, this assertion was proved by Massera ⁽⁵⁾.

If in the definition of V -monotonicity one can take as $V(t, x, y)$ the quadratic form $(A(x - y), x - y)$, where A is a nonsingular symmetric ...

symmetric matrix, then we shall say that $F(x, t)$ is **monotone with respect to A** , i.e.

$$W(t, x, y) \equiv (A(x - y), F(x, t) - F(y, t)) < 0.$$

If $\overline{W}(t, x, y) \leq W(x, y)$, then $F(x, t)$ is uniformly monotone with respect to A .)

Theorem 6. Let $F(x, t)$ satisfy the local theorems on existence and uniqueness and be uniformly monotone with respect to A , where A has invariant subspaces H_k^+ of positive eigenvalues and H_{n-k}^- of negative eigenvalues. $\|F(x, t)\|$ is bounded on every bounded set X , uniformly in t .

- 1) $\|F(x, t)\| \leq L(\|x\|)$, $L(r) > 0$ continuous;

- 2) $(Ax, F(x, t)) \leq -\varepsilon(\|x\|)$ for sufficiently large $\|x\| > \bar{R} > 0$, $\varepsilon(r) > 0$ continuous.

Let

$$\int^{\infty} \frac{\varepsilon(r)}{L(r)} dr = +\infty.$$

Then system (1) has a unique solution $\Phi(t)$ bounded on the whole axis I . If $F(x, t)$ is ω -periodic or almost periodic with modulus M_{H_0} , then $\Phi(t)$ will be a periodic or almost-periodic solution satisfying (7).

The behavior of the remaining solutions is given by the following theorem:

Theorem 7. For each fixed t_0 one can indicate in R^n two manifolds M_k^+ (M_{n-k}^-), which are graphs of continuous mappings from H_k^+ to H_{n-k}^- (from H_{n-k}^- to H_k^+). M_k^+ lies in the domain $A^+\{(Ax, x) > 0\}$, and M_{n-k}^- lies in the domain $A^-\{(Ax, x) < 0\}$.

If at the moment t_0 the difference $x(t_0) - \Phi(t_0) \in M_k^+$ (M_{n-k}^-), then

$$\|x(t) - \Phi(t)\| \rightarrow 0$$

as $t \rightarrow +\infty$ ($t \rightarrow -\infty$), and is not bounded to the left (to the right). If $x(t_0) - \Phi(t_0) \notin M_k^+$ (M_{n-k}^-), then $\|x(t) - \Phi(t)\|$ grows without bound in both directions in t . If all solutions can be continued to the whole axis I

$$\left(\text{for this it is sufficient, for example, that } \int^{\infty} \frac{dr}{L(r)} = \infty \right),$$

then it is sufficient to consider initial values only from the hyperplane $t = 0$; thereby all solutions are covered.

Theorem 8. Let $\|F(x, t)\|$ in any bounded domain in X be bounded uniformly in t , and suppose that for sufficiently large $\|x\|$ the estimate holds

- 1) $\|F(x, t)\| \leq L(\|x\|)\|x\|$, where $L(\|x\|)$ may grow without bound with the growth of $\|x\|$;
- 2) $(A(x - y), F(x, t) - F(y, t)) \leq -\varepsilon(r)\|x - y\|^2$ ($\|x\|$ and $\|y\| \leq r$),
 $\varepsilon(r) > 0$ ($r > 0$), $\varepsilon(r)r \rightarrow +\infty$ ($r \rightarrow +\infty$).

Let

$$\int^{\infty} \frac{\varepsilon(r)r}{L(r)} dr = +\infty.$$

Then all conditions of Theorem 6 are fulfilled, and all assertions of Theorems 6 and 7 are valid.

Theorem 8 generalizes the results of A. I. Perov for the finite-dimensional case.

I express my gratitude to my scientific adviser Prof. V. V. Nemytskii and Prof. B. M. Levitan for their attention.

Moscow State University
named after M. V. Lomonosov

Received
26 II 1966

REFERENCES

1. V. I. Zubov, *Oscillations in Nonlinear and Controlled Systems*, L., 1962.
2. A. Halanay, Com. Ac. Rep. Pop. Rom., 6, No. 1 (1956).
3. T. Yoshizawa, Arch. Rat. Mech. Anal., 17, No. 2 (1964).
4. N. N. Krasovskii, *Some Problems of the Theory of Stability of Motion*, M., 1959.
5. I. G. Malkin, *Theory of Stability of Motion*, M., 1952, p. 300.
6. A. I. Perov, DAN, 132, No. 3 (1960).
7. G. Seifert, J. Math. Anal. and Applic., 10, No. 2 (1965).
8. V. M. Cherevko, DAN, 165, No. 2 (1965).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.