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Abstract

Full Text

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ON THE MEAN VALUES OF THE SUPPORT FUNCTION

1. Let G be a convex body in n -dimensional Euclidean space; $h(x, \nu)$ the distance from the point $x \in G$ to the supporting plane to G with outward normal ν ; Ω the unit sphere (the set of all ν); χ_n its area. Define the mean values of the function $h(x, \nu)$

$$h_k(x) = \left[\frac{1}{\chi_n} \int_{\Omega} h^{-k}(x, \nu) d\nu \right]^{-1/k}, \quad k \neq 0, \quad (1)$$

$$h_0(x) = \exp \frac{1}{\chi_n} \int_{\Omega} \ln h(x, \nu) d\nu. \quad (2)$$

These functions h_k deserve attention not only because interesting geometric problems are connected with them, but also because they majorize, up to a factor, the solutions $u(x)$ of second-order equations of a fairly general form under $u|_{\partial G} = 0$ ^(1,2). In ⁽²⁾, for linear equations the majorants are given in the form $Hh_k(x)$ with one or another k , $0 \leq k \leq n$, where H is expressed in terms of certain norms of the coefficients (and of the solution itself, if uniqueness conditions are not satisfied). Further, in ⁽³⁾ it is proved that the majorants with h_0 and h_n are sharp. Thus, for example, the function h_0 admits a definition through solutions of elliptic linear second-order equations: $h_0(s) = \sup |u(x)|$, where the supremum is taken over the solutions (with $u|_{\partial G} = 0$) of all equations for which $H = 1$.

2. Let us list some of the simplest properties of the function h_k .

(2.1) If $G \subset G'$, $G \neq G'$, then for every k and every $x \in G$

$$h_k(x) < h'_k(x).$$

(2.2) If $k > l$, then $h_k(x) < h_l(x)$, except in the case when G is a ball and x is its center. (This follows from the well-known property of means; see, for example, ⁽⁷⁾.)

(2.3) If $k > -1$, then everywhere inside G , $d^2 h_k < 0$, while for $k < -1$, $d^2 h_k > 0$. For the proof we note that

$$h(x, \nu) = h(x_0, \nu) + (x_0 - x)\nu.$$

Substituting this in (1) or (2), it is easy to find $d^2 h_k$ and to verify the asserted statement. In addition, we find that $h_{-1}(x) = \text{const}$.

(2.4) As is known, if from the point x in each direction ν one lays off a segment of length $r(x, \nu) = h^{-1}(x, \nu)$, then one obtains the convex body G^x , polar to G with respect to the point x (with respect to the unit ball with center x). Therefore, for $k \neq 0$,

$$h_k(x) = \left[\frac{1}{\chi_n} \int_{\Omega} r^k(x, \nu) d\nu \right]^{-1/k}. \quad (3)$$

Hence it is easy to conclude that if k is an integer, $1 \leq k \leq n$, then

$$h_k(x) = \tau_k^{1/k} \hat{V}_k^{-1/k}(x), \quad (4)$$

where τ_k is the volume of the k -dimensional unit ball, and $\hat{V}_k(x)$ is the mean value of the k -dimensional volumes of the sections of the body G^x by k -dimensional planes passing through x ; in particular, $\hat{V}_n(x)$ is the volume of G^x . (The mean is understood here and below as the arithmetic mean in the sense of the natural measure in the set of ...

set of planes of the given dimension passing through a fixed point.)

What has been said makes it possible, among other things, to find $h_n(x)$ when the polar body G^x is simply determined. For example, the body polar to an ellipsoid is an ellipsoid. Its volume is found from elementary geometric considerations. Thus we find that for an ellipsoid

$$h_n(x) = (a_1, \dots, a_n)^{1/n} (1 - \rho^2(x))^{(n+1)/2n},$$

where a_i are the semiaxes, and $\rho(x)$ is the ratio of the distance from the center to x to the radius in the same direction.

3. Let us indicate estimates for $H_k \neq \max h_k$ in terms of the following quantities: V_m —the greatest of the volumes of the m -dimensional sections of the body G ; W_m —the mean value of the volumes of its m -dimensional projections; w_m —the least of these volumes. In particular, $V_n = W_n = w_n$ is the volume of G , V_1 is the diameter, W_1 is the mean width. (Some inequalities between these quantities and the relation of W_m to the integral curvatures can be found in ^(4, 5). In particular

$$W_{n-1} = \frac{\tau_{n-1}}{\nu_n} S,$$

where S is the area of ∂G , $\tau_m^{-1/m} W_m^{1/m} \geq \tau_{m+1}^{-1/(m+1)} W_{m+1}^{1/(m+1)}$.) We shall denote by α_m , etc., positive numbers depending only on m and the dimension n .

(3.1) For every k , obviously, $H_k < V_1$. On the other hand, for $k < 1$, $H_k > \alpha_k V_1$, but for $k \geq 1$ such a lower estimate is impossible. (It suffices to note that G contains a segment of length V_1 , and therefore, if H'_k is $\max h_k$ for such a segment, then $H_k > H'_k$. Computing H'_k , we obtain what was asserted.)

(3.2) For $k > -1$, $H_k < \frac{1}{2} W_1$, except in the case when G is a ball. By definition,

$$W_1 = \frac{1}{\alpha_n} \int (h(x, \nu) + h(x, -\nu)) d\nu = \frac{2}{\alpha_n} \int h(x, \nu) d\nu = 2h_{-1}(x).$$

Therefore $h_{-1}(x) = \frac{1}{2} W_1$, and the assertion follows from (2.2). On the other hand, it also follows from (2.2) that for $k < -1$, $h_k(x) > h_{-1}(x)$, again except in the case of a ball. Therefore, excluding this case, for $k < -1$

$$\min h_k(x) > \frac{1}{2} W_1$$

(cf. this with (2.3)).

(3.3)

$$\alpha_n V_n^{1/n} \geq H_n \geq \beta_n V_n^{1/n}.$$

By virtue of (4) $H_n = \tau_n^{1/n} \widetilde{V}_n^{-1/n}$, where $\widetilde{V}_n = \min_x \widetilde{V}_n(x)$ is the least of the volumes of the polar bodies G^x . Therefore (3.3) is equivalent to

$$\lambda_n \leq V_n \widetilde{V}_n \leq \mu_n, \quad \lambda_n = \tau_n \alpha_n^{-n}, \quad \mu_n = \tau_n \beta_n^{-n}. \quad (5)$$

The fact that such inequalities for $V_n \widetilde{V}_n$ hold is known and is proved simply. From the known properties of mutually polar bodies there follows the affine invariance of the product $V_n \widetilde{V}_n$. Therefore it is enough to consider bodies of volume $V_n = 1$ contained in a given cube. Then it is obvious that the product $V_n \widetilde{V}_n$ attains finite and positive maximum and minimum values, and this is precisely (5). It is not difficult also to obtain some values for λ_n, μ_n , but the question of the best values remains open (see, for example, (6), where $\lambda_n = 4^n (n!)^{-2}$ is given). It is probable that the greatest value of $V_n \widetilde{V}_n$ is attained for an ellipsoid, and the least for a simplex, to which, accordingly, the best values of μ_n, λ_n correspond.

(3.4) If $1 \leq k < n$ and l is the integer part of k , then

$$\alpha'_k W_l^{1/l} \geq H_k \geq \beta'_k W_{l+1}^{1/(l+1)},$$

$$\alpha''_k V_l^{1/l} \geq H_k \geq \beta''_k V_{l+1}^{1/(l+1)} \quad \text{and} \quad H_k \geq \gamma_k w_l,$$

but lower estimates in terms of W_l, V_l are impossible also for $k = l$.

Let us prove the first inequality: $H_k \leq \alpha_k W_l^{1/l}$. Since for $k \geq l$, $H_k \leq H_l$, one may put $k = l$. Let E be an k -dimensional plane, pro-

passing through the center of the sphere Ω ; x_E is the projection of the point x onto E ; G_E is the projection of G . For $v \in \Omega \cap E$, obviously, $h(x, v) = h(x_E, v)$, that is, the distance from x_E to the supporting plane of G_E with normal v . Therefore, if we put

$$h_k(x, E) = \left[\frac{1}{\chi_k} \int_{\Omega \cap E} h^{-k}(x, v) dv \right]^{-1/k}, \quad (6)$$

then $h_k(x, E)$ will be nothing other than the function h_k for G_E . Hence, according to (3.3), we conclude that $h_k(x, E) = h_k(x_E, E) \leq \alpha_k V_k^{1/k}(E)$, where $V_k(E)$ is the volume of G_E .

If the measure in the set $\mathcal{E}$ of planes E is normalized so that $\text{mes } \mathcal{E} = 1$, then for every integrable $f(v)$ we have

$$\frac{1}{\chi_n} \int_{\Omega} f(v) dv = \int_{\mathcal{E}} \left(\frac{1}{\chi_k} \int_{\Omega \cap E} f(v) dv \right) dE.$$

Applying this equality to $f(v) = h^{-k}(x, v)$ and using (6), we obtain

$$h_k^{-k}(x) = \frac{1}{\chi_n} \int_{\Omega} h^{-k}(x, v) dv = \int_{\mathcal{E}} \left(\frac{1}{\chi_k} \int_{\Omega \cap E} h^{-k}(x, v) dv \right) dE = \int_{\mathcal{E}} h_k^{-k}(x, E) dE,$$

and since $h_k(x, E) \leq \alpha_k V_k^{1/k}$, i.e. $h_k^{-k}(x, E) \geq \alpha_k^{-k} V_k^{-1}(E)$, it follows that

$$h_k^{-k}(x) \geq \alpha_k^{-k} \int_{\mathcal{E}} V_k^{-1}(E) dE \geq \alpha_k^{-k} \left(\int_{\mathcal{E}} V_k(E) dE \right)^{-1} = \alpha_k^{-k} W_k^{-1},$$

where we have used the well-known inequality for means and the definition of the quantity W_k . Thus, for every $x \in G$ we have $h_k(x) \leq \alpha_k W_k^{1/k}$, as was required to prove.

We omit the proofs of the other inequalities (3.4). The inequalities with W and V follow from one another, since $\delta_m \geq W_m V_m^{-1} \geq \delta'_m$; moreover, $W_{m+1} \geq \delta''_m w_m$.

4. We now indicate functions estimating $h_k(x)$, $0 < k \leq n$.

(4.1) Let G_m be the body symmetric to G with respect to the point $x \in G$, and let $V(x)$ be the volume of the body $H = G \cap G_x$. It turns out that for $h_n(x)$

$$\bar{\alpha}_n V_n^{1/n}(x) \geq h_n(x) \geq \beta_n V_n^{1/n}(x), \quad (7)$$

where, incidentally, β_n is the same as in (3.3). (From the left inequality (7) it is not difficult to conclude that if $\bar{V}(x)$ is the smallest of the volumes cut off from G by arbitrary planes passing through x , then $h_n(x) \leq \alpha_n 2^{1/n} \bar{V}^{1/n}(x)$.)

Let us prove (7). Let $\bar{h}_n$ be the function h_n for the body H . Obviously, $h_n(x) \geq \bar{h}_n(x)$, and by (3.3) $\bar{h}_n(x) \geq \beta_n V^{1/n}(x)$, whence the right-hand inequality (7) follows.

For the proof of the left-hand inequality we use the lemma:

Lemma. If K is a convex body, K_x is the body symmetric to it with respect to the point $x \in K$, and $\bar{K}$ is the convex hull of $K \cup K_x$, then

$$V(K) \geq \gamma_n V(\bar{K}). \quad (8)$$

For the proof, observe that, by the affine invariance of the ratio of volumes, one may assume $V(K) = 1$ and that K is contained in some ball. On the set of bodies K satisfying these conditions, for all $x \in K$, the volumes $V(\bar{K})$ are bounded. If γ_n^{-1} is their least upper bound, then (8) holds.

We apply this lemma to $K = \tilde{G}$, where $\tilde{G}$ is the body polar to G with respect to the point x . Then $K_x = \tilde{G}_x$ is the body polar to G_x . From the properties of polarity we conclude that $\bar{K}$ is the body $\tilde{H}$, polar to $H = G \cap G_x$ (because under polarity the common points of bodies correspond to planes not intersecting their polars). Therefore (8) gives:

$$V(\tilde{G}) \geq \gamma_n V(\tilde{H}). \quad (9)$$

But, as a consequence of (4), $\bar{V}(\tilde{G}) = \tau_n \bar{h}_n^{-n}(x)$, $\bar{V}(\tilde{H}) = \tau_n \bar{h}_m^{-n}(x)$. Therefore, from (9) it follows that $\bar{h}_n(x) \geq \gamma_n^{1/n} \bar{h}_m(x)$, and since, by (3.3), $\bar{h}_n(x) \leq \bar{a}_n V^{1/n}(x)$, we obtain the left-hand inequality (7). (The best values of the constants in (7), (8) are unknown. The best γ_n is the minimum of $V(\bar{K}) : V(K)$; it is probably attained when K is a simplex and x is one of its vertices.)

(4.2) Let $h(x)$ be the distance from $x \in G$ to ∂G , i.e.

$$h(x) = \min_v h(x, v).$$

For every $k > 0$ and $\leq n$,

$$h_k(x) \leq h^{1-n/k}(x) h_n^{n/k}(x) \leq \bar{a}_n^{n/k} h^{1-n/k}(x) V^{1/k}(x), \quad (10)$$

where $V(x)$, $\bar{a}_n$ are the same as in (4.1).

The second inequality is a consequence of (7). The first is obtained as follows: in view of (1) and the fact that $h(x) \leq h(x, v)$,

$$h_k(x) = h^{1-n/k}(x) \left[\frac{1}{\chi_n} \int_{\Omega} h^{k-n}(x) h^{-k}(x, v) dv \right]^{-1/k} \leq h^{1-n/k}(x) h_n^{n/k}(x).$$

Analogously one obtains the lower estimate with $H(x) = \max h(x, v)$:

$$h_k(x) \geq H^{1-n/k}(x) h_n^{n/k}(x) \geq \beta_n^{n/k} H^{1-n/k}(x) V^{1/k}(x).$$

Estimate (10), in particular, determines when and with what rate $h_k(x) \rightarrow 0$, if $x \rightarrow x_0 \in \partial G$. If at x_0 the body G is, so to speak, sufficiently convex, then $V(x) \rightarrow 0$ so rapidly that also $h^{1-n/k}(x) V^{1/k}(x) \rightarrow 0$. For example, from (10) it is easy to conclude that for $k > n - 1$

$$h_k(x) \leq A h^{1-(n-1)/k}(x), \quad A = A(n, k, V_1).$$

If, however, $k = n - 1$, then $h_k(x)$ vanishes at all points of ∂G , except those lying inside flat faces.

Further, if G is contained in a paraboloid of degree l with vertex $x_0 \in \partial G$, then

$$h_k(x) \leq A h^q(x), \quad q = 1 - (n - 1)(l - 1)/kl.$$

If G contains a similar paraboloid with vertex x_0 , then for points on its axis near x_0 ,

$$h_k(x) \leq A_1 h^q(x).$$

This shows that estimate (10) in this case gives the correct order of magnitude of $h_k(x)$.

(4.3) If the point $x_0 \in \partial G$ is conical, then

$$h_0(x) \leq A |x - x_0|^{\omega/\chi_n},$$

where ω is the solid angle of the normals to the supporting planes at x_0 . (The proof consists in estimating $h_0(x)$ for a cone.) If, however, $x_0 \in \partial G$ is the vertex of a paraboloid of degree $l > 1$ contained in G , then $h_0(x_0) > 0$. For $k < 0$, $h_k(x) > 0$ everywhere on ∂G . Let us also note that on the surface of the unit sphere

$$\ln h_0(x) = (-1)^{n-1} \left(\ln 2 + \sum_{m=1}^{n-2} \frac{(-1)^m}{m} \right).$$

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REFERENCES

- ¹ A. D. Aleksandrov, *Siberian Mathematical Journal*, No. 3 (1966).
- ² A. D. Aleksandrov, *Vestnik Leningrad University*, No. 1 (1966).
- ³ A. D. Aleksandrov, *Vestnik Leningrad University*, No. 7 (1966).
- ⁴ H. Busemann, *Convex Surfaces*, Moscow, 1964.
- ⁵ T. Bonnesen, W. Fenchel, *Theorie der konvexen Körper*, Berlin, 1934.
- ⁶ R. P. Bambah, *Proc. Camb. Phil. Soc.*, **51**, Part 2, 377 (1955).
- ⁷ G. G. Khardil, D. E. Littlewood, G. Polya, *Inequalities*, Moscow, 1948.

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