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Abstract

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MATHEMATICS

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ON THE DENSITY HYPOTHESIS OF E. BOMBIERI

(Presented by Academician Yu. V. Linnik on 9 IV 1966)

In a recent work ⁽¹⁾, Bombieri, generalizing Yu. V. Linnik' s "large sieve" method ⁽²⁾, proved an elegant density theorem, a simple consequence of which is the following form of the "averaged" law of distribution of prime numbers in arithmetic progressions:

$$\sum_{D \leq x^{1/2}/(\ln x)^B} \max_{(l,D)=1} \left| \pi(x, D, L) - \frac{\text{li } x}{\varphi(D)} \right| = O \left(\frac{x}{\ln^A x} \right) \quad (1)$$

for any arbitrarily large but fixed A , and for some $B = B(A)$. Estimate (1) replaces the extended Riemann hypothesis in well-known additive problems ^(3,4). (An estimate close to (1) was obtained by A. I. Vinogradov in ⁽⁵⁾, by another method and independently; there an interesting application to the generalized divisor problem of Hooley-Linnik is also given.)

Many problems of this type were first solved by Yu. V. Linnik' s dispersion method ⁽⁶⁾, which in essence uses the achievements of modern algebraic geometry and whose natural sphere of action is formed by binary problems not derivable from the extended Riemann hypothesis, for example,

$$N = Q(x, y) + P(p_1, p_2),$$

where $Q(x, y)$, $P(x, y)$ are positive definite quadratic forms; p_1, p_2 are prime numbers, $p_1 \leq N^\alpha$, $p_2 \leq N^{1-\alpha}$.

In a somewhat roughened form Bombieri' s density theorem has the form

$$\sum_{D \leq X} \sum_{\chi_D} N(a, T, \chi'_D) \ll X^{8(1-\alpha)/(3-2\alpha)+\varepsilon} T^c, \quad (2)$$

where $N(a, T, \chi)$ denotes the number of zeros of the Dirichlet L -series $L(s, \chi)$ in the rectangle $a \leq \sigma \leq 1$, $|t| \leq T$, with $s = \sigma + it$, $T \geq 2$, $a \geq 1/2$; ε is arbitrarily small but fixed; χ'_D is a primitive character mod D .

In the same work ⁽¹⁾, Bombieri formulates the hypothesis

$$\sum_{D \leq X} \sum_{\chi'_D} N(a, T, \chi'_D) \ll X^{4(1-\alpha)+\varepsilon} T^{1+\varepsilon}. \quad (3)$$

Results of type (2), (3) obviously make it possible to conclude that, for almost all moduli D , the quasi-Riemann hypothesis holds (for all χ'_D , $L(s, \chi'_D)$ have no zeros in the rectangle $\sigma > \gamma_0$, $|t| \leq D^\varepsilon$, $\gamma_0 < 1$ an absolute constant). This, in turn, makes it possible to obtain a power saving in the asymptotic formula for the moments of the number of ideal classes of a quadratic field.

The following two inequalities constitute a strengthening of Bombieri's estimate (2), useful for the applications described.

Theorem 1. *The inequalities hold*

$$\sum_{D \leq X} \sum_{\chi'_D} N(a, T, \chi'_D) \ll X^{10(1-\alpha)/(3-\alpha)+\varepsilon} T^c, \quad (4)$$

$$\sum_{D \leq X} \sum_{\chi'_D} N(\alpha, T, \chi'_D) \ll X^{10/4(1-\alpha)+\varepsilon} T^c \quad (5)$$

uniformly for all α , $1/2 \leq \alpha \leq 1$.

Inequality (4) is obtained by Gabriel's method (see (7), p. 238) and the following estimate of the eighth moment of L -series on the critical line:

$$\sum_{D \leq X} \sum_{\chi_D \neq \chi_0} |L(1/2 + it, \chi)|^8 \ll X^{2+\varepsilon} (|t| + 2)^c \quad (6)$$

(an obvious consequence of Theorem 3 of Bombieri's paper (1) and the truncated functional equation for Dirichlet L -series).

Inequality (5) is proved according to the classical scheme (see (8)). Here the main role is played by Burgess' s estimate (9) for the modulus of the L -series on the line $\sigma = 1/2$.

An obvious consequence of inequality (4) of Theorem 1 is

Theorem 2. For all moduli D in the interval $[1, X]$, with the exception of at most $X^{1-\delta}$, all L -series corresponding to primitive characters mod D have no zeros in the region $|\sigma| > 7/9 + \delta$, $|t| \leq D^\gamma$ ($\varepsilon = \varepsilon(\delta, \gamma)$).

Using Theorem 2 according to the scheme of paper (10), we obtain the following theorem.

Theorem 3. Let $h(-D)$ denote the number of classes of purely radical quadratic forms of determinant $-D$, $D > 0$. Then for any fixed k one has

$$\sum_{D \leq N} h^k(-D) = \frac{2^{k+1} r(k)}{\pi^k (k+2)} N^{(k+2)/2} \{1 + O(N^{-\xi})\}, \quad (7)$$

where

$$r(k) = \sum_{\substack{n=1 \\ n \equiv 1 \pmod{2}}}^{\infty} \frac{\varphi(n) \tau_k(n^2)}{n^3}, \quad \xi \text{ is any constant smaller than}$$

$$(\sqrt{129} - 9)/2 \approx 0.18.$$

A substantial advance in this question would now be a proof of (7) with $\xi = 1/2 + \varepsilon$ (Bombieri's hypothesis (3) gives $\xi = 1/5 + \varepsilon$).

In conclusion we formulate the hypothesis.

Hypothesis. The inequality

$$\sum_{D \leq X} \max_{\chi'_D} N(\alpha, T, \chi'_D) \ll X^{2(1-\alpha)+\varepsilon} T^c$$

holds uniformly for $1/2 \leq \alpha \leq 1$.

Let us note that a consequence of this hypothesis is the absence (for almost all moduli D) of zeros of the functions $L(s, \chi'_D)$ in the rectangle $\sigma > 1/2 + \varepsilon$; $|t| \leq D^\gamma$, as well as the possibility of choosing $\xi = 1/3 + \varepsilon$ in Theorem 3.

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