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Abstract

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MATHEMATICS

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θ -SPACES AND PERFECT IRREDUCIBLE MAPPINGS OF TOPOLOGICAL SPACES

(Presented by Academician P. S. Aleksandrov on 25 VII 1966)

In the present paper the notion of a θ -space is introduced, generalizing the notion of a proximity space compatible with a completely regular space.

At the same time, θ -spaces exist on every regular space, whereas proximity spaces are compatible only with completely regular spaces. A connection is given between θ -spaces on a regular space X and bicomcompact extensions of all its perfect irreducible preimages. In this connection, a theorem of Yu. M. Smirnov⁽⁴⁾ on the one-to-one correspondence between proximity spaces compatible with a given completely regular space X and bicomcompact extensions of the space X is generalized. The paper also introduces the notion of a θ -mapping of θ -spaces and studies its connection with mappings of bicomcompact extensions of perfect irreducible preimages of topological spaces.

Let X be a regular space. We shall say that a θ -proximity is given on X if for any two subsets $A \subset X$ and $B \subset X$, either $A\theta B$ or $A\bar{\theta}B$ is specified and the following axioms are satisfied:

I. $A\theta B \Rightarrow B\theta A$.

II. $A\bar{\theta}B_i, i = 1, 2 \Leftrightarrow A\bar{\theta}(B_1 \cup B_2)$.

III. $\emptyset\bar{\theta}X$.

IV. $\{x\}\theta A \Rightarrow x \in [A]$.

V. $A\bar{\theta}B \Rightarrow$ there exists such a $C = \langle [C] \rangle \supset A$ that $A\bar{\theta}(X \setminus [C])$ and $C\bar{\theta}B^*$.

Axioms I-IV coincide with the corresponding axioms of a proximity space compatible with the given topological space, and it is easy to see that axiom V is a weakening of the normality axiom of a proximity space compatible with a topological space.

An example of a θ -proximity on a regular space X is the following relation: $A\bar{\theta}_a B \Leftrightarrow$ there exist disjoint neighborhoods of the sets A and B . It is easy to verify that all the axioms are satisfied. It will be shown below that the θ -proximity θ_a is maximal.

Theorem 1. Let $f : Z \rightarrow X$ be a perfect irreducible mapping of a completely regular space Z onto a space X , and let bZ be a bicomact extension of the space Z . Then the bicomactum bZ generates on X the following θ -proximity:

$$A\bar{\theta}B \iff [f^{-1}A]_{bZ} \cap [f^{-1}B]_{bZ} = \emptyset.$$

We shall call a regular space X with a θ -proximity given on it a θ -space.

* By $\langle D \rangle$ is denoted the interior of the set D .

The main result of the paper is

Theorem 2. Every θ -space on a regular space X determines a completely regular space X_θ , a perfect irreducible projection $\pi_{X_\theta} : X_\theta \rightarrow X$ of the space X_θ onto X , and a bicomact extension $b_\theta X_\theta$, which generates the given θ -proximity.

Remark. The space $b_\theta X_\theta$ is the space of maximal centered systems $\tau = \{H\}$ of open subsets H of the space X with the following θ -regularity condition: for every $H \in \tau$ there exists an $H' \in \tau$ such that $H' \theta (X \setminus [H])^*$.

Theorem 3. Let $f_1 : Z_1 \rightarrow X$ and $f_2 : Z_2 \rightarrow X$ be perfect irreducible mappings of completely regular spaces Z_1 and Z_2 onto X . If the bicomact extensions $b_1 Z_1$ and $b_2 Z_2$ generate on X the same θ -space, then there exists a homeomorphism $g : b_1 Z_1 \rightarrow b_2 Z_2$ such that $gZ_1 = Z_2$ and $f_1 = f_2 g$.

Corollary 1. There exists a one-to-one correspondence between the partially ordered set of all θ -spaces on a regular space X and the set of all pairs $(bZ, f : Z \rightarrow X)$, where bZ is a bicomact extension of the space Z and f is a perfect irreducible mapping onto X .*

Corollary 2. There exists a maximal θ -space on a regular space X .

Corollary 3. For any two perfect irreducible mappings $f_1 : Z_1 \rightarrow X$ and $f_2 : Z_2 \rightarrow X$ onto X there exists a perfect irreducible mapping $f : Z \rightarrow X$ onto X and perfect irreducible mappings $g_1 : Z \rightarrow Z_1$ onto Z_1 and $g_2 : Z \rightarrow Z_2$ onto Z_2 such that $f_1 g_1 = f = f_2 g_2$, and moreover the mapping f is minimal in the sense that mappings f' , g'_1 and g'_2 with the same relations factor through the mappings f , g_1 and g_2 , respectively**.

S. Iliadis proved [2] that the space B of maximal centered systems of open sets of a regular space X is the Čech extension of the absolute of the space X . A maximal centered system of open sets in a regular space X obviously satisfies the θ -regularity condition for the θ -proximity θ_a . Hence, by the remark to Theorem 2 and Corollary 1, θ_a is the maximal θ -proximity.

Definition. A perfect irreducible mapping $f : X \rightarrow Y$ of a θ -space X onto a θ -space Y will be called a θ -mapping if $A, B \subset Y$ and

$$\overline{A} \theta B \Rightarrow \langle f^{-1}[A] \rangle \bar{\theta} \langle f^{-1}[B] \rangle.$$

Theorem 4. Every θ -mapping $f : X \rightarrow Y$ generates a perfect irreducible mapping $f_\theta : b_\theta X_\theta \rightarrow b_\theta Y_\theta$ such that $f_\theta X_\theta = Y_\theta$ and $f\pi_{X\theta} = \pi_{Y\theta}f_\theta$.

In accordance with (3), we shall call a multivalued mapping $f : X \rightarrow Y$ of θ -spaces a *multivalued θ -mapping* if there exists a θ -space Z and θ -mappings $f_X : Z \rightarrow X$, $f_Y : Z \rightarrow Y$ such that $f = f_Y f_X^{-1}$.

* The notion of a centered system of open sets with a certain regularity condition was introduced by P. S. Aleksandrov [1] in the construction of the maximal bicomact extension.

** The order in this set is analogous to the order in the set of proximity spaces compatible with the given space.

*** Two pairs $(bZ, f : Z \rightarrow X)$ and $(b'Z', f' : Z' \rightarrow X)$ are identified if there exists a homeomorphism $g : bZ \rightarrow b'Z'$ such that $gZ = Z'$ and $f = f'g$. $(bZ, f) \geq (b'Z', f') \iff$ there exists a perfect irreducible mapping $g : bZ \rightarrow b'Z'$ such that $gZ = Z'$ and $f = f'g$. It is easy to verify that pairs (bZ, f) and $(b'Z', f')$ for which both inequalities $(bZ, f) \leq (b'Z', f')$ and $(bZ, f) \geq (b'Z', f')$ hold are identified.

**** A mapping $f' : Z' \rightarrow X$ factors through a mapping $f : Z \rightarrow X$ if there exists a mapping $g : Z' \rightarrow Z$ such that $f' = fg$.

Theorem 5. Every multivalued θ -mapping $f : X \xleftarrow{f_X} Z \xrightarrow{f_Y} Y$ generates a multivalued perfect irreducible mapping

$$f_\theta : b_\theta X_\theta \xleftarrow{\varphi} b_\theta Z_\theta \xrightarrow{\psi} b_\theta Y_\theta,$$

such that $\varphi^{-1}X_\theta = \psi^{-1}Y_\theta$ and $f\pi_{X\theta} = \pi_{Y\theta}f_\theta$.

Theorem 6. Let θ -spaces X and Y be given. Then for every multivalued perfect irreducible mapping $f_\theta : b_\theta X_\theta \xleftarrow{\varphi} B \xrightarrow{\psi} b_\theta Y_\theta$ satisfying $\varphi^{-1}X_\theta = \psi^{-1}Y_\theta$, there exists a multivalued θ -mapping $f : X \xleftarrow{f_X} Z \xrightarrow{f_Y} Y$ such that

$$B = b_\theta Z_\theta, \quad f\pi_{X\theta} = \pi_{Y\theta}f_\theta, \quad (*)$$

and, among all multivalued θ -mappings satisfying condition (*), there exists a maximal one in the factorization sense.

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