

ON THE CONTINUITY OF MAPPINGS AND TRANSLATIONS

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Abstract

Full Text

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MATHEMATICS

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ON THE CONTINUITY OF MAPPINGS AND TRANSLATIONS

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1. Let a topological space R be given, in which a complete system of neighborhoods is defined (1). We shall say that a complete system of neighborhoods is defined for every set $M \subset R$, if for M a system $\Sigma(M)$ of sets open in R is given, possessing the following properties: a) for every set $U \in \Sigma(M)$ the inclusion $M \subset U$ holds; b) for every pair of sets U and V from $\Sigma(M)$ there exists a set $W \in \Sigma(M)$ such that $W \subset U \cap V$; c) for every set $U \in \Sigma(M)$ and every set $M' \subset \overline{M}$ there exists a set $V \in \Sigma(M')$ such that $V \subset U$; d) if $M' \subset M$, then for every set $U \in \Sigma(M)$ there is a set $U' \in \Sigma(M')$ such that $U' \subset U$; e) if a sequence of points convergent in R is almost entirely contained in an arbitrary set U of the system $\Sigma(M)$, then its limit is a point of contact of the set M ; f) if the set M consists of one point, then $\Sigma(M)$ coincides with the complete system of neighborhoods of this point. The totality of all neighborhoods of all sets in R will be called the complete system of generalized neighborhoods of the space R . It is assumed that all topological spaces occurring in the article are, once and for all, equipped with neighborhoods and generalized neighborhoods; moreover, in metric spaces, as neighborhoods there are taken, as usual, all possible spheres of points, and as generalized neighborhoods—all possible "spheres" of sets.
2. A point sequence $x_k \in R$, $k = 1, 2, \dots$, will be called bounded in R if there exists a neighborhood of the space R containing almost all elements of this sequence. We shall assume that sequences not bounded in R can be distinguished from one another by the character of unboundedness or by order of growth.

We shall distinguish point sequences in R by types. Every point sequence convergent in R will be assigned to the zeroth type, a bounded one—to the second type, an unbounded sequence with growth not exceeding a certain order—to the fourth type, and, finally, an arbitrary point sequence in R —to the sixth type.

3. Let a sequence of sets M_k , $k = 1, 2, \dots$, be given in R . A sequence of sets M_k^* , $k = 1, 2, \dots$, will be called a daughter sequence for the sequence M_k ,

if for every k the inclusion $M_k^* \subset M_k$ is satisfied.

We shall say that a sequence of sets M_k , $k = 1, 2, \dots$, converges in R of the i -th type to the set $M \subset R$ ($i = 0, 2, 4, 6$), if for every neighborhood V of the set M and every sequence of points x_k , $k = 1, 2, \dots$, of the i -th type and daughter for the sequence M_k , under the condition that the set of such sequences is nonempty, there is a number k_0 such that from the inequality $k \geq k_0$ there follows the inclusion $x_k \in V$.

We shall say that a sequence of sets M_k , $k = 1, 2, \dots$, converges in R of the $(i+1)$ -st type to the set $M \subset R$ ($i = 0, 2, 4, 6$), if it converges in R to the set M of the i -th type and for every point $x \in M$

there is a sequence of points x_k , $k = 1, 2, \dots$, subordinate to the sequence M_k , for which the point x is a limit point.

If a sequence of sets M_k , $k = 1, 2, \dots$, converges in R of the seventh type to a set M , then it converges in R of the seventh type also to the set $\bar{M}$, the latter being uniquely determined and naturally called the **limit of the seventh type** of the sequence M_k .

Let σ_i denote the type of convergence in R of a sequence of sets to some set ($i = 0, 1, 2, 3, 4, 5, 6, 7$). It is easy to see that the following implications hold:

$$\sigma_6 \rightarrow \sigma_0, \quad \sigma_6 \rightarrow \sigma_2, \quad \sigma_6 \rightarrow \sigma_4;$$

$$\sigma_7 \rightarrow \sigma_1, \quad \sigma_7 \rightarrow \sigma_3, \quad \sigma_7 \rightarrow \sigma_5;$$

$$\sigma_1 \rightarrow \sigma_0, \quad \sigma_3 \rightarrow \sigma_2, \quad \sigma_5 \rightarrow \sigma_4, \quad \sigma_7 \rightarrow \sigma_6.$$

4. Let two spaces* R_1 and R_2 be given. We shall say that a translation B from R_1 into R_2 is defined if a rule is specified by which a set $P_B \subset R_1$ is determined, and to each element x of P_B there is assigned a definite set η of elements of R_2 . In this case we shall write $\eta = Bx$. As x ranges over P_B , $\eta \subset R_2$. The set P_B will be called the **domain of definition** of the translation B . The union of all sets $\eta = Bx$, as x ranges over P_B , will be denoted by Q_B and called the **range** of the translation B . It is clear that B is a translation of P_B onto Q_B : $\eta = Bx$, x ranges over P_B , and Q_B is the union of the sets $\eta = Bx$, x ranges over P_B .

If the set $\eta = Bx$ consists of one point, then x is called a **point of single-valuedness** of the translation B . If all points of P_B are points of single-valuedness for B , then B is a mapping of P_B onto Q_B .

If a translation B of the space P_B onto Q_B is defined, then thereby also defined is the translation which assigns to each point $y \in Q_B$ the set ξ of all elements x of P_B for which $y \in Bx$. This translation is called the **inverse** of B and is

denoted by the symbol B^{-1} . If the inverse of the translation B is a mapping, then the translation B is called **one-to-one**; otherwise, **many-to-one**.

Assign to each point $x \in P_B$ a definite subset B^*x of the set Bx . As a result there arises a translation B^* of the space P_B into Q_B , which we shall call **subordinate** to the translation B .

We shall say that a family of mappings B_α **exhausts the translation** B if, for every α , the mapping B_α is subordinate to B , and for every $x \in P_B$ the equality

$$Bx = \bigcup_{\alpha} B_{\alpha}x$$

holds.

5. Let R_1 and R_2 be topological spaces. A translation B from R_1 into R_2 will be called **complete** if, for every $x \in P_B$, the set Bx is closed. A translation B will be called **regular** if, for every $x \in P_B$, the set Bx is bounded. A translation B from R_1 into R_2 will be called **continuous at the point** $x = x_0$ **of the i -th type** ($x_0 \in P_B$, $i = 0, 1, 2, 3, 4, 5, 6, 7$) if, for every sequence of points $x_k \in P_B$, $k = 1, 2, \dots$, converging in R_1 to the point x_0 , the sequence Bx_k , $k = 1, 2, \dots$, converges in R_2 of the i -th type to Bx_0 . A translation B from R_1 into R_2 will be called **continuous of the i -th type** if at each point of the set P_B it is continuous of the i -th type.

Theorem 1. *If a translation B from R_1 into R_2 is continuous of the i -th type ($i = 0, 1, 2, 3, 4, 5, 6, 7$), then every restriction of the translation B is continuous of the i -th type.*

Theorem 2. *If a mapping B of the space R_1 into R_2 is continuous and, for every sequence of points $y_k \in Q_B$, $k = 1, 2, \dots$, converging in R_2 to a point $y_0 \in Q_B$, for the sequence of sets $B^{-1}y_k$, $k = 1, 2, \dots$,*

* In this section, by a space is meant a nonempty fixed set of some elements (?).

there exists a daughter sequence of points of zero type, then the translation B^{-1} is discontinuous of zero type at the point y_0 .

Theorem 3. If the mapping B of the space R_1 into R_2 is discontinuous and open, then the translation B^{-1} is discontinuous of the first type.

Theorem 4. If the mapping B of the space R_1 into R_2 is open, then the image of every closed set is a closed set.

6. Let R be a metric space with metric ρ . Consider the functional

$$\Psi_x y = \rho(x, y), \quad y \text{ ranges over } \overline{Q},$$

where Q is a nonempty subset of R and $x \in R$.

Theorem 5. If the set Q is dense in R , or if every sphere of the space R is compact, then the functional Ψ_x has a minimum; moreover, the set M_x of elements minimizing this functional is a full regular translation of the space R into $\bar{Q}$, discontinuous of the sixth type.

7. Let R_1 and R_2 be metric spaces with metrics ρ_1 and ρ_2 , respectively. A family of mappings B_α from R_1 into R_2 , defined on one and the same set $P \subset R_1$, will be called **uniformly discontinuous** at a point $x_0 \in P$ if, for every $\varepsilon > 0$, there exists a neighborhood $V_{x_0} \subset R_1$ of the point x_0 such that the inclusion

$$B_\alpha(V_{x_0} \cap P) \subset s(B_\alpha x_0, \varepsilon) \cap B_\alpha P.$$

holds.

Theorem 6. Let a translation B from R_1 into R_2 be given, and let a family of mappings B_α exhausting it be defined. Then, if the family B_α is uniformly discontinuous at each point of the set P_B , the translation B is discontinuous of the seventh type.

Denote by R_3 the topological product of the spaces R_1 and R_2 , with metric $\rho_3 = \max(\rho_1, \rho_2)$. If K_1 and K_2 are compact-in-themselves subsets of the spaces R_1 and R_2 , respectively, then K_3 , equal to the topological product of K_1 by K_2 , will be a compact-in-itself subset of the space R_3 , and conversely.

Theorem 7. Let a real continuous functional $F(x, y)$ be given on a compact-in-itself set K_3 of the space R_3 , where x ranges over K_1 and y ranges over K_2 . We shall regard $F(x, y)$ as a functional only of y for fixed x : $F_x y = F(x, y)$, y ranges over K_2 . Then the set N_x of elements minimizing the functional $F_x y$, where y ranges over K_2 , is a full, regular translation of K_1 into K_2 , discontinuous of the sixth type.

8. **Theorem 8.** Let a translation B be given from the topological space R_1 onto the topological space R_3 . Suppose, moreover, that a translation T of the subspace P_B onto the space R_2 and a translation S of the space R_2 onto the space R_3 are given such that $B = ST$.

A. Then, if the space R_2 is topological, the translation S is discontinuous of the zero (first) type, and the translation T is full and discontinuous of the zero (first) type, then the translation B is discontinuous of the zero (first) type.

B. Then, if the space R_2 is metric and each of its closed spheres is compact, the translation T is full and regular, and the translations S and T are discontinuous of the sixth type, then the translation B is discontinuous of the sixth type.

Remark. Theorem 8B is also true in the case when the set Q_B is a proper part of the space R_3 .

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CITED LITERATURE

1. L. S. Pontryagin, *Continuous Groups*, Moscow, 1954.
2. L. Collatz, *Funktionalanalysis und numerische Matematik*, Berlin, 1964.

Note: Figure translations are in progress. See original paper for figures.

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