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Abstract

Full Text

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MATHEMATICS

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A NECESSARY AND SUFFICIENT CONDITION FOR STABILITY IN THE UNIFORM METRIC OF SYSTEMS OF DIFFERENCE EQUATIONS

(Presented by Academician S. L. Sobolev, 14 V 1966)

We consider the problem with initial data

$$\sum_{|l| \leq k} A_l^1 u_{j+l}^{n+1} = \sum_{|l| \leq k} A_l^0 u_{j+l}^n, \quad \mathbf{u}_j^0 = \mathbf{r}_j, \quad |r_j(p)| < \infty. \quad (1)$$

Here A_l^1 and A_l^0 are matrices with constant coefficients of dimension (q, q) , and $\mathbf{u}_j^n$ is a vector of dimension q . Stability in the uniform metric, in C , is investigated.

Definition. Problem (1) is **stable in C** if, uniformly in n , the relation

$$\sup_{j, 1 \leq r \leq q} |u_j^n(r)| \leq c \sup_{j, 1 \leq r \leq q} |u_j^0(r)|$$

holds.

Here and below c, c_1, c_2 are positive and do not depend on n ; by $u_j^n(r)$ we denote the components of the vector $\mathbf{u}_j^n$.

After the transformation

$$\mathbf{v}^n = \sum_j \mathbf{u}_j^n z^{-j}$$

problem (1) takes the form

$$\mathbf{v}^n = \{(A^1(z))^{-1} A^0(z)\}^n \mathbf{v}^0 = C^n(z) \mathbf{v}^0.$$

If the matrix $A^1(z)$ is nonsingular on the unit circle $|z| = 1$ (denote this circle by D_1), then the matrix $C(z)$ is defined everywhere on D_1 . Moreover, the elements of $C(z)$ are ratios of polynomials.

In order for system (1) to be stable in C , it is necessary and sufficient that, uniformly in n , the Fourier coefficients of the elements of the n -th power of the characteristic matrix $C(z)$ be bounded in the l_1 metric; namely, if

$$\Gamma_j^n = \frac{1}{2\pi} \int_0^{2\pi} C^n(e^{i\varphi}) e^{ij\varphi} d\varphi,$$

then it is necessary and sufficient that, uniformly in n , the relation

$$\sup_{1 \leq r_1, r_2 \leq q} \sum_j |\Gamma_j^n(r_1, r_2)| \leq c$$

hold.

Theorem. In order that system (1) be stable in C , it is necessary and sufficient that:

- 1) system (1) satisfy the necessary and sufficient condition for stability in L_2 , formulated by V. Ya. Urm (¹);
- 2) each eigenvalue of the matrix $C(z)$ satisfy the necessary and sufficient condition for stability in C , formulated by me (^{2,3}) for the characteristic function in the case of a single equation. For systems of difference equations that are stable in L_2 , but unstable in the uniform metric, we obtain estimates, sharp in order, for the growth of

$$\sup_{1 \leq r_1, r_2 \leq q} \sum_j |\Gamma_j^n(r_1, r_2)|.$$

Below the second part of the theorem is refined, and in conclusion an example is given of a system that is stable in L_2 , but unstable in C .

One of the conditions for stability of system (1) in L_2 is the requirement that all eigenvalues of the matrix $C(z)$ on D_1 do not exceed unity in modulus. We note that the eigenvalues of the matrix $C(z)$ are algebraic functions. Hence, and from the condition of stability in L_2 , it follows that each eigenvalue of $C(z)$ belongs to one of the following three types:

I. The eigenvalue $\lambda(z)$ is identically equal in modulus to unity on D_1 , but $\lambda(z)$ is not a power of z .

II. $\lambda(z) = e^{ib_0} z^{l/m}$.

III. The eigenvalue is less than unity in modulus everywhere on D_1 , except for a finite number of points $e^{i\varphi_1}, \dots, e^{i\varphi_s}, \dots, e^{i\varphi_t}$, where $|\lambda(e^{i\varphi_s})| = 1$.

In a neighborhood of each φ_s the expansion is valid

$$\lambda(e^{i\varphi}) = \exp \left\{ ib_0 + ib_1(\varphi - \varphi_s) + i \sum_{l=l_1}^{2p} b_l(\varphi - \varphi_s)^l - \alpha(\varphi - \varphi_s)^{2p} + \sum_{l=1}^{\infty} c_l(\varphi - \varphi_s)^{2p+l/m} \right\}. \quad (2)$$

The coefficients b_l , $\alpha > 0$ are real, and l, p, m are integers.

We refine the second part of the theorem. In order that system (1), stable in L_2 , be stable in C , it is necessary and sufficient that all eigenvalues of the matrix $C(z)$ be either of type II or of type III, and for each eigenvalue of type III and all φ_s where this eigenvalue is equal to unity in modulus, the relation

$$l_1 \geq 2p.$$

must hold in the expansion (2).

If there is at least one eigenvalue of type I, then system (1) is unstable in C . The estimate

$$c_1 n^{1/2} \leq \sup_{1 \leq r_1, r_2 \leq q} \sum_j |\Gamma_j^n(r_1, r_2)| \leq c_2 n^{1/2}$$

is valid.

If there are no eigenvalues of type I, but there is at least one eigenvalue of type III for which, for some φ_s , the relation

$$l_1 < 2p,$$

holds in the expansion (2), then there is no stability in C . The estimate

$$c_1 n^{1/2-l_1/4p} \leq \sup_{1 \leq r_1, r_2 \leq q} \sum_j |\Gamma_j^n(r_1, r_2)| \leq c_2 n^{1/2-l_1/4p}.$$

is valid.

Here $l_1/4p$ is the least of all possible values $l_1/4p$ corresponding to the various eigenvalues of type III and the various φ_s .

Example. The system

$$u_j^{n+1} - u_j^n - \frac{1}{2} \frac{\tau}{h} (v_{j+1}^n - v_{j-1}^n) - \frac{1}{2} \frac{\tau^2}{h^2} (u_{j+1}^n - 2u_j^n + u_{j-1}^n) = 0,$$

$$v_j^{n+1} - v_j^n - \frac{1}{2} \frac{\tau}{h} (u_{j+1}^n - u_{j-1}^n) - \frac{1}{2} \frac{\tau^2}{h^2} (v_{j+1}^n - 2v_j^n + v_{j-1}^n) = 0$$

is stable in L_2 for $\tau/h < 1$. In a neighborhood of $\varphi = 0$, the eigenvalues of the characteristic matrix of this system have the form

$$\lambda_{1,2}(\varphi) = \exp \left\{ \pm i \frac{\tau}{h} \varphi \mp i \frac{1}{6} \frac{\tau}{h} \left(1 - \frac{\tau^2}{h^2} \right) \varphi^3 - \frac{1}{8} \frac{\tau^2}{h^2} \left(1 - \frac{\tau^2}{h^2} \right) \varphi^4 + O(\varphi^5) \right\};$$

here $l_1 = 3$, $2p = 4$, so that $l_1 < 2p$, and therefore there is no stability in C :

$$\sup_{1 \leq r_1, r_2 \leq 2} \sum_j |\Gamma_j^n(r_1, r_2)| \sim (n^{1/8}).$$

Remark. The problem with initial data for a multilayer scheme reduces (4) to a two-layer problem of the form (1), so that the results formulated here simultaneously resolve the question of the stability of multilayer schemes.

For example, the study of the stability of the three-layer scheme

$$u_j^{n+1} - u_j^{n-1} = \frac{\tau}{h} (u_{j+1}^n - u_{j-1}^n)$$

reduces to the study of the stability of the system:

$$u_j^{n+1} - v_j^n = \frac{\tau}{h} (u_{j+1}^n - u_{j-1}^n),$$

$$v_j^{n+1} = u_j^n.$$

The latter, for $\tau/h < 1$, is stable in L_2 , but unstable in the uniform metric,

$$\sup_{1 \leq r_1, r_2 \leq 2} \sum_j |\Gamma_j^n(r_1, r_2)| \succ (n^{1/2}).$$

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