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Abstract

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MATHEMATICS

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ON MOTIONS CLOSE TO A DOUBLY ASYMPTOTIC MOTION

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In the present paper the structure of the motions of a multidimensional dynamical system in a sufficiently small neighborhood of a motion doubly asymptotic to a certain saddle periodic motion is clarified. The dynamical system is assumed to be described by ordinary differential equations with right-hand sides that are twice continuously differentiable.

We intersect the phase curve Γ_0 corresponding to the periodic motion orthogonally at some point O by a secant hyperplane S . Let Γ be a phase trajectory doubly asymptotic to Γ_0 . This trajectory Γ , which we shall call *homoclinic*, intersects, as $t \rightarrow +\infty$ and, respectively, as $t \rightarrow -\infty$, the secant surface S in some sufficiently small neighborhood of the point O in two sequences of points $M^0, M^1, M^2, \dots$ and $N^0, N^1, N^2, \dots$, asymptotically approaching the point O .

Denote by T the point mapping of the secant surface S into itself, generated by phase trajectories close to Γ_0 . This point mapping is defined in some neighborhood of the point O , is one-to-one, and is twice continuously differentiable. The fixed point O of the mapping T is of saddle type, so that its characteristic roots split into two groups $\lambda_1, \lambda_2, \dots, \lambda_m$ and $\nu_1, \nu_2, \dots, \nu_n$ such that $|\lambda_i| < 1$ and $|\nu_j| > 1$. Denote by π^+ and π^- the sets of points of the secant plane S asymptotically approaching the point O under successive applications of the transformation T and, respectively, T^{-1} . The sequences of points $M^0, M^1, M^2, \dots$ and $N^0, N^1, N^2, \dots$ lie respectively on π^+ and π^- . As is known, π^+ and π^- are smooth manifolds of dimensions m and n , respectively, intersecting at the point O . If these manifolds are taken as coordinate surfaces, then the point mapping T can be written in the form

$$\bar{x} = \lambda x + f(x, y)x, \quad \bar{y} = \nu y + g(x, y)y, \quad (1)$$

where x and y are respectively m - and n -dimensional vectors, with $y = 0$ on π^+ , and $x = 0$ on π^- ; λ and ν are constant matrices with eigenvalues $\lambda_1, \dots, \lambda_m$ and $\nu_1, \dots, \nu_n$, respectively; $f(x, y)$ and $g(x, y)$ are matrices that vanish at the point O ($x = 0, y = 0$). A doubly asymptotic (homoclinic) phase curve issuing

from the point $N^0(0, y^0)$ again intersects the surface S , after some finite time, at the point $M^0(x^0, 0)$. Thus, in a neighborhood of the point N^0 a point map is defined—

* Homoclinic motion was discovered by A. Poincaré ⁽¹⁾. Subsequently the so-called homoclinic point was studied in a number of works ^(2–6), etc. Birkhoff ^(2, 3) showed that in the two-dimensional case, in a neighborhood of a homoclinic point, there is an infinite set of multiple fixed points (see also ⁽⁴⁾). A proof of the existence of an infinite number of periodic motions in a neighborhood of a homoclinic motion in the multidimensional case was given by L. P. Shilnikov (unpublished).

the mapping L , which takes the point N^0 into the point M^0 . Let the mapping L_ν take π^- into $\bar{\pi}^-$. The transformed manifold $\bar{\pi}^-$ contains the point M^0 . Suppose that $\bar{\pi}^-$ and π^+ intersect at the point M^0 roughly, i.e., the tangent planes to $\bar{\pi}^-$ and π^+ at the point M^0 , of dimensions n and m , intersect in the single point M^0 . Under these assumptions the following theorem holds.

Theorem 1. There exist such $\bar{N}$ and $\delta > 0$ that to every sequence of integers

$$\dots, \gamma_{-j}, \dots, \gamma_{-1}, \gamma_0, \gamma_1, \dots, \gamma_j, \dots, \quad (2)$$

larger than $N \geq \bar{N}$, there corresponds a unique sequence of points

$$\dots, (x_{-2}, y_{-2}), (x_{-1}, y_{-1}), (x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, \quad (3)$$

belonging to $\bar{\delta}(M^0)$, such that

$$(x_j, y_j) = LT^{\gamma_j}(x_{j-1}, y_{j-1}). \quad (4)$$

The points of the sequences (3) lie in $\delta(M^0)$, and moreover $\delta \rightarrow 0$ as $N \rightarrow \infty$. To a periodic sequence (2) there corresponds a periodic sequence of points (3) with the same period, which in turn corresponds to a periodic motion of saddle type. To sequences (2) differing only in indices j greater than k , there correspond sequences (3) for which the points (x_0, y_0) differ in norm by less than δ , and moreover $\delta \rightarrow 0$ as $k \rightarrow \infty$.

Corollary*. In every arbitrarily small neighborhood of a homoclinic motion there is an infinite set of periodic motions of all possible multiplicities and of nonperiodic motions that are recurrent and Poisson-stable. In a sufficiently small neighborhood Γ the periodic motions form a countable set. Through an arbitrarily small neighborhood of each of the periodic motions, in turn, there passes an uncountable set of other periodic motions. For each of the periodic motions there is an uncountable set of doubly asymptotic motions.

Proof. Introduce the auxiliary mapping (see also (7)) $\tilde{T}$ as the mapping taking the point $(x, \bar{u})$ into the point $(\bar{x}, y)$, if the mapping T takes the point (x, y) into the point $(\bar{x}, \bar{y})$. It can be shown that, in the case when the points M^0 and N^0 lie in a sufficiently small neighborhood of the point O (which may be assumed fulfilled), the auxiliary mappings $\tilde{T}^\gamma$, for $\gamma > N_1$, are defined, single-valued, and contracting with a contraction coefficient tending to zero as $\gamma \rightarrow \infty$ in some neighborhood ω of the point (x^0, y^0) . The mapping L

$$\bar{x} = h(x, y), \quad \bar{y} = k(x, y) \quad (5)$$

takes the point $N^0(0, y^0)$ into the point $M^0(x^0, 0)$, and, consequently, for any sufficiently small neighborhood $\delta(M^0)$ one can specify a neighborhood $\delta'(N^0)$ that is mapped into it. By virtue of the assumption made on the roughness of the intersection of $\bar{\pi}^-$ and π^+ at the point N^0 , $K'_y(x, y)$ is an invertible matrix. Because of this, the auxiliary mapping $\tilde{L}$ is defined in some neighborhood of the point $(0, 0)$ and takes the point $(0, 0)$ into the point (x^0, y^0) .

Let a τ -neighborhood of the point $(0, 0)$ be mapped by the transformation L inside $\omega(x^0, y^0)$. Since the contracting mapping $\tilde{T}^\gamma$ maps the point (x^0, y^0)

* We note that phase curves lying entirely in a small neighborhood of the homoclinic curve Γ , which again and again, both as t increases and as t decreases, intersect the cutting surface S in a sufficiently small neighborhood of the point M^0 , correspond to some sequence (2) and (3) with γ_s greater than some number that increases as the mentioned neighborhood of the homoclinic curve decreases; and to every sequence (2) with $\gamma_s > N \geq \bar{N}$ there corresponds a phase curve lying in a sufficiently small neighborhood of Γ , for which the corresponding points (3) are successive points of intersection with S in $\delta(M^0)$.

maps into the point $(x^\gamma, y^{-\gamma}) \rightarrow 0$ as $\gamma \rightarrow \infty$, then for $\gamma > N_2 \geq N_1$ the mapping T transforms $\omega(x^0, y^0)$ into $\tau(0, 0)$. Let the point mapping $\tilde{L}T^\gamma$ be contracting for $\gamma > N_3 > N_2$; then for $\gamma > N_3$ this point mapping transforms $\omega(x^0, y^0)$ into itself and has in this neighborhood a unique fixed point. To the fixed point of the mapping $\tilde{L}T^\gamma$ there corresponds one-to-one the fixed point of the mapping $LT^{\gamma*}$. Therefore, in some neighborhood of the point M_0 , the mapping LT^γ has a fixed point, and this fixed point tends to the point M^0 as γ increases. The fixed point of the mapping LT^γ is at the same time a fixed point of the contracting mapping $\tilde{L}T^\gamma$, whose contraction coefficient decreases without bound as $\gamma \rightarrow \infty$. Let, for $\gamma > \bar{N} \geq N_3$, the contracting mapping $\tilde{L}T^\gamma$ take the σ -neighborhood of the point M^0 into itself. Let $\gamma_{l-1} > \bar{N}$ and $\gamma_l > \bar{N}$. The contracting mapping $LT^{\gamma_{l-1}}LT^{\gamma_l}$ maps $\sigma(M^0)$ into itself and has in it a unique fixed point, to which the fixed point of the mapping $LT^{\gamma_{l-1}}LT^{\gamma_l}$ corresponds one-to-one. Now, in an analogous way, one can establish the existence of a fixed point for the contracting mapping

$$LT^{\gamma_{l-2}} \widetilde{LT^{\gamma_{l-1}}} L^{\gamma} T$$

and, consequently, also for the mapping

$$LT^{\gamma_{l-2}} LT^{\gamma_{l-1}} LT^{\gamma_l},$$

and so on. These fixed points are of saddle type, since the auxiliary mappings are contracting.

From the last assertion of the theorem follows the uniqueness of the sequence (3) corresponding to an arbitrary sequence (2). The last assertion of the theorem follows from Theorem 3 of [7] with the conditions (13) slightly generalized. In the case of a nonperiodic sequence (2), the required sequence (3) can be found as the limit for periodic sequences (3) corresponding to periodic sequences with period $\gamma_{-j}, \dots, \gamma_0, \dots, \gamma_j$ as $j \rightarrow \infty$. Indeed, let (x_0^j, y_0^j) be the points corresponding to these sequences, and let (x_0, y_0) be their limiting point. The sequence (3), obtained from the point (x_0, y_0) by means of the relations (4), is the desired one. For the proof it is necessary only to make sure that all points of the sequence obtained in this way do not leave the δ -neighborhood of the point M^0 , if all the corresponding periodic sequences do not leave the δ' -neighborhood and $\delta' < \delta$. This last assertion follows directly from the continuity of the mappings LT^{γ_s} . The proof just given allows us to formulate the following lemma.

Lemma. Let each of the point mappings T_j ($j = 1, 2, \dots, p$) have a fixed point M_j , and let the auxiliary mappings $\widehat{T}_j$ be contracting and transform some region $G = G_x \oplus G_y$, containing the points M_j , into itself. Then any mapping of the form $T_{j_g} T_{j_{g-1}} \dots T_{j_1}$, where $1 \leq j_1 \leq p, \dots, 1 \leq j_g \leq p$, has a fixed point in the region G .

Theorem 2. To each of the sequences of the form

$$\gamma_{-i}, \dots, \gamma_j, \infty; \quad \infty, \gamma_{-i}, \dots, \gamma_j, \dots; \quad \infty, \gamma_{-i}, \dots, \gamma_j, \infty \quad (6)$$

with $\gamma_s > \bar{N}$ there corresponds a unique point (x_0, y_0) such that the points (x_s, y_s) , defined according to (4), lie in $\delta(M^0)$, and, moreover, in the first case $(x_{-i-1}, y_{-i-1}) \subset \pi^+$, in the second case $(x_j, y_j) \subset \pi^-$, and in the third both take place.

Corollary 1. In an arbitrarily small neighborhood of a homoclinic curve there exists an infinite set of other homoclinic curves

* Indeed, from the relations $LT^{\gamma}(x, y) = L(\bar{x}, \bar{y}) = (x, y)$ it follows that $LT^{\gamma}(x, y) = L(\bar{x}, y) = (x, \bar{y})$, and conversely.

(with the same ω - and α -limit periodic motion). In a sufficiently small neighborhood of a homoclinic curve there is a countable set of homoclinic curves.

Corollary 2. To every motion lying, for all t , in a sufficiently small neighborhood Γ , there corresponds uniquely either a sequence (2), or (6), and to each such sequence with $\gamma_s > \bar{N}$ there corresponds a unique motion belonging to a sufficiently small neighborhood Γ .

Proof. Let (x_0^γ, y_0^γ) be the point which, according to Theorem 1, corresponds to the periodic sequence with the period obtained from the corresponding sequence (6) by replacing ∞ by γ . As $\gamma \rightarrow \infty$, the limiting point (x_0, y_0) of the points (x_0^γ, y_0^γ) is the required one. Uniqueness is proved in the same way as in the case of Theorem 1.

Denote by $\sigma(\gamma_{-i}, \dots, \gamma_j)$ the set of points (x_0, y_0) of a sufficiently small neighborhood $\delta(M^0)$ for which the points (x_s, y_s) , defined according to (4) for $s = -i, -1, \dots, j$, lie in $\delta^0(M^0)$.

Theorem 3. There exists an $\bar{N}$ such that, for $\gamma_s > N \geq \bar{N}$, all the sets $\sigma(\gamma_{-i}, \dots, \gamma_j)$ are nonempty domains lying in $\delta'(M^0)$, with $\delta' \rightarrow 0$ as $N \rightarrow \infty$. The domains $\sigma(\gamma_{-i}, \dots, \gamma_j)$ and $\sigma(\gamma'_{-i'}, \dots, \gamma'_{j'})$ do not intersect if, for some common index s , $\gamma'_s \neq \gamma_s$. If $j' \geq j$, $i \leq i'$, and $\gamma'_s = \gamma_s$ for all common values of s , then

$$\sigma(\gamma_{-i}, \gamma_j) \subset \sigma(\gamma'_{-i'}, \dots, \gamma'_{j'}).$$

The measure of the domain $\sigma(\gamma_{-i}, \dots, \gamma_j)$ does not exceed

$$C\gamma_q^{-i} + \dots + \gamma_j,$$

where C is a constant and $q < 1$.

Corollary. The set of motions lying, for all t , in a sufficiently small neighborhood of a homoclinic curve has measure zero.

Proof. The set $\sigma = \sigma(\bar{\gamma}_{-i}, \dots, \bar{\gamma}_j)$ is nonempty, since it contains the points (x_0, y_0) corresponding to all possible sequences (3) and (6) for which $\gamma_{-i} = \bar{\gamma}_{-i}, \dots, \gamma_j = \bar{\gamma}_j$. By definition,

$$LT^{\bar{\gamma}_j} \dots LT^{\bar{\gamma}_{-i}} \sigma \subset \delta(M^0) \quad \text{and} \quad T^{-\gamma_0} L^{-1} \dots T^{-\gamma_{-i}} L^{-1} \sigma \subset \delta(M^0).$$

For sufficiently large γ_s , the mappings LT^{γ_s} and $T^{-\gamma_s} L^{-1}$ are stretching, with stretching coefficient not less than $q^{-\gamma_s}$ ($q < 1$), along mutually complementary subspaces; from this follows the last assertion of the theorem.

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Note: Figure translations are in progress. See original paper for figures.

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