

# A DISCONTINUOUS PROBLEM OF LINEAR CONJUGATION FOR ANALYTIC FUNCTIONS ON RIEMANN SURFACES

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**Abstract**

**Full Text**

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*MATHEMATICS*

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# A DISCONTINUOUS PROBLEM OF LINEAR CONJUGATION FOR ANALYTIC FUNCTIONS ON RIEMANN SURFACES

*(Presented by Academician I. N. Vekua on 8 IV 1966)*

Let on a closed Riemann surface  $R$  of genus  $\rho$  there be given a contour  $\Gamma$ , consisting of a finite number of mutually nonintersecting closed oriented curves satisfying the Lyapunov condition. Suppose, further, that measurable functions  $G$  and  $g$  are given on  $\Gamma$ . It is required to determine a function  $\Phi$ , holomorphic in  $R - \Gamma$ , belonging to the class  $H_p$ ,  $p \geq 1^*$  on each component of  $R - \Gamma$ , whose boundary values almost everywhere on  $\Gamma$  satisfy the equality

$$\Phi^+ = G\Phi^- + g. \quad (1)$$

This problem has been well studied in the planar case under various restrictions on  $G$  and  $g$  (see, for example, <sup>(2-5)</sup>). On a Riemann surface it was studied for  $G$  and  $g$  satisfying the Hölder condition <sup>(6,7)</sup> and when  $G$  has discontinuities of the first kind <sup>(8,9)</sup>.

In the present note a method is given, analogous to the well-known alternating method of Schwarz, which makes it possible to investigate problem (1) on a Riemann surface under restrictions on  $G$  and  $g$  adopted in the papers <sup>(2-5)</sup>.

Let on  $R$  there be given two domains  $K_0$  and  $K_1$ ,  $\overline{K_1} \subset K_0$ , bounded by analytic curves and conformally equivalent to planar circular rings. Suppose, further, that in  $K_0 - K_1$  a harmonic function  $u_0$  is given.

**Theorem 1.** The condition

$$\int_{\partial K_0} du_0^* = 0 \quad (2)$$

is necessary and sufficient for the existence of a function  $u$  with the following properties:

A.  $u$  is single-valued and harmonic on  $R - K_1$ .

B.  $u - u_0$  is the restriction to  $K_0 - K_1$  of a function single-valued and harmonic in  $K_0$ .

The necessity of condition (2) is obvious.

Sufficiency. Consider sequences of harmonic functions  $\{u_n\}$  and  $\{v_n\}$ , defined in the domains  $R - K_1$  and  $K_0$ , respectively, by the boundary conditions

$$u_n = v_{n-1} + u_0 \quad \text{on } \partial(R - K_1), \quad (3)$$

$$v_n = u_n - u_0 \quad \text{on } \partial K_0 \quad (4)$$

$$(n = 1, 2, \dots; v_0 \equiv 0).$$

Map the domain  $K_0$  by means of an analytic function  $\tau$  one-to-one and conformally onto the planar annulus  $1 < |z| < r$  and choose two numbers  $r_1, r_2$ ,  $1 < r_1 < r_2 < r$ , with  $\ln r_1 = \ln r/r_2$ , so that  $\tau^{-1}(r_2 < |z| < r_1)$  and  $\tau^{-1}(r_2 < |z| < r)$  lie in different components of  $K_0 - K_1$ . On the basis—

\* For the definition of the Hardy classes  $H_p$  on finite Riemann surfaces, see <sup>(1)</sup>.

of equalities (2), (3), and (4) it is easy to show that

$$\int_0^{2\pi} [v_n(e^{i\varphi}) - v_{n-1}(e^{i\varphi}) + v_n(re^{i\varphi}) - v_{n-1}(re^{i\varphi})] d\varphi = 0. \quad (5)$$

Consequently,

$$\min_{K_0} (v_n - v_{n-1}) \leq 0, \quad \max_{K_0} (v_n - v_{n-1}) \geq 0.$$

Denote by  $s(w, M)$  the oscillation of the function  $w$  on the set  $M$ . From the last inequalities it follows that

$$|v_n - v_{n-1}| \leq s(v_n - v_{n-1}, K_0). \quad (6)$$

According to Lemma 3 of [10],

$$s(v_n - v_{n-1}, K_1) \leq q s(v_n - v_{n-1}, K_0), \quad (7)$$

where  $q < 1$  is a constant depending only on  $K_0$  and  $K_1$ . From (3), (4), and (7) we obtain

$$s(v_{n+1} - v_n, K_0) \leq q^n s(v_1, K_0). \quad (8)$$

The last inequality, together with inequality (6), entails the uniform convergence of the sequence  $\{v_n\}$  in  $K_0$ . From the convergence of  $\{v_n\}$  there follows the convergence of  $\{u_n\}$  in  $R - K_1$ . Denote  $v = \lim_{n \rightarrow \infty} v_n$ ,  $u = \lim_{n \rightarrow \infty} u_n$ . Passing to the limit in equalities (3) and (4), we obtain that the functions  $v$  and  $u - u_0$  coincide in  $K_0 - K_1$ . The theorem is proved.

Let now a function  $h \in L_p$ ,  $p \geq 1$ , be given on  $\Gamma$ . Consider the so-called Sokhotski problem: it is required to determine on  $R - \Gamma$  a function  $F \in H_p$  whose boundary values satisfy on  $\Gamma$  the condition

$$F^+ - F^- = h. \quad (9)$$

If a solution of the problem exists, then, as is easy to see,  $\int_{\Gamma} h dw = 0$  for any differential of the first kind  $dw$ .

Choose for each  $\Gamma_j$  a doubly connected neighborhood  $K_j$ , map it by means of an analytic function  $\tau_j$  onto a plane annulus, and construct in the plane a piecewise-holomorphic function  $f_j \in H_p$  with respect to the boundary condition prescribed on  $\gamma_j = \tau_j(\Gamma_j)$ ,

$$f_j^- - f_j^+ = h \circ \tau_j^{-1}. \quad (10)$$

The function  $F_j = f_j \circ \tau_j$ , defined in  $K_j - \Gamma_j$ , is holomorphic, belongs to  $H_p$ , and satisfies condition (9) on  $\Gamma_j$ . According to Theorem 1, there exists a function  $u_j$ , harmonic in  $R - \Gamma_j$ , such that  $u_j - \operatorname{Re} F_j$  is the restriction to  $K_j - \Gamma_j$  of a single-valued function harmonic in  $K_j$ . On each component of  $R - \Gamma_j$  construct the function

$$F'_j = u_j + iu_j^*. \quad (11)$$

The restriction of any branch  $F'_j$  to  $K_j - \Gamma_j$  belongs to  $H_p$  and satisfies on  $\Gamma_j$  the condition

$$F_j'^+ - F_j'^- = h + iC_j, \quad (12)$$

where  $C_j$  is a real constant depending on the chosen branch of  $F'_j$ .

Form the differential  $dF = \sum_{j=1}^m dF_j$  and compute its periods on  $R - \Gamma$ . The calculation of the periods is carried out with the aid of the bilinear relation, taking (12) into account. The following assertion holds: for any cycle  $l \subset R - \Gamma$  there exists a differential of the first kind  $dZ_l$ , depending only on  $l$ , such that

$$\int_l dF = \operatorname{Im} \int_{\Gamma} h dZ_l.$$

Let  $\int_l dF = 0$  for any  $l \subset R - \Gamma$ . Then the analytic function

$$F = \int dF$$

is single-valued on  $R - \Gamma$ , belongs to  $H_p$ , and satisfies the boundary-condition

$$F^+ - F^- = h + iB, \quad (13)$$

where  $B$  is a real function, equal to a constant on each  $\Gamma_j$ . The value  $B$  on a nondividing component of the contour  $\Gamma$  can be made zero by adding a constant to  $F$ . The value  $B_j$  of the function  $B$  on a nondividing  $\Gamma_j$  can be computed. To this end one must take an Abelian differential of the first kind  $dZ_{\Gamma_j}$ , whose real part has a single nonzero period, equal to one along  $\Gamma_j$ .

From Cauchy's theorem it follows that

$$B_j = \text{Im} \int_{\Gamma} h dZ_{\Gamma_j}.$$

All that has been said confirms the validity of the following theorem.

**Theorem 2.** *For the solvability of the Sokhotski problem it is necessary and sufficient that*

$$\int_{\Gamma} h dw_{\nu} = 0 \quad (\nu = 1, 2, \dots, \rho),$$

where  $dw_1, dw_2, \dots, dw_{\rho}$  is a basis of the space of Abelian differentials of the first kind.

As mentioned above, the investigation of problem (1) is carried out under the assumptions adopted in papers <sup>(2-5)</sup>. In the planar case it follows from these assumptions that:

- I.  $\ln G \in L_{p_1}$ ,  $p_1 \geq 1$ .
- II.  $G = X^+(t)/X^-(t)$ ,  $t \in \Gamma$ , where  $X^{\pm}(z) \in H_p$ ,  $1/X^{\pm}(z) \in H_q$ ,  $q = p/(p-1)$ , and  $X(z)$  has neither zeros nor poles outside  $\Gamma$ , except at the point at infinity, where the order of  $X(z)$  is equal to the index of the problem.
- III. The general solution of problem (1) is given by the formula  $\Phi = XF + XP$ , where  $F$  is a solution of the Sokhotski problem determined by the jump  $g/X^+$ , and  $P$  is a polynomial.

Thus, without formulating the restrictions imposed on  $G$  and  $g$ , we shall assume that the functions  $G \circ \tau_j^{-1}$  and  $g \circ \tau_j^{-1}$  satisfy conditions I-III on  $\gamma_j$ .

Let us first consider the homogeneous problem

$$\Phi^+ = G\Phi^-. \quad (14)$$

For the given  $G$  one can construct a function  $G_0$ , satisfying the Hölder condition on  $\Gamma$ , such that

$$\begin{aligned} \frac{1}{2\pi i} \int_{\Gamma_j} d \ln G_0 &= \text{ind}_{\gamma_j}(G \circ \tau_j^{-1}) \quad (j = 1, 2, \dots, m), \\ \frac{1}{2\pi i} \int_{\Gamma} \ln G_0 dw_\nu &= \frac{1}{2\pi i} \int_{\Gamma} \ln G dw_\nu \quad (\nu = 1, 2, \dots, \rho). \end{aligned}$$

The functions  $\frac{G}{G_0} \circ \tau_j^{-1}$  satisfy conditions I, II and

$$\text{ind}_{\gamma_j} \left( \frac{G}{G_0} \circ \tau_j^{-1} \right) = 0.$$

Taking this into account, on the basis of Theorems 1 and 2 it is not difficult to establish that the homogeneous problem with coefficient  $G/G_0$  is solvable. Its solution  $X_0$  is single-valued and nowhere vanishes outside  $\Gamma$ . Dividing both sides of (14) by  $X_0$ , we arrive at the boundary-value problem defined by the boundary condition

$$\Phi_0^+ = G_0 \Phi_0^-. \quad (15)$$

If (15) is solvable in the class  $H_p$ , then its solution is automatically continuously extendable to  $\Gamma$ . Problems (14) and (15) are equivalent. The general solution of problem (14) is given by the formula  $\Phi = X_0 \Phi_0$ . The number of linearly independent solutions of problem (15), and hence of problem (14), depends on the numbers

$$\frac{1}{2\pi i} \int_{\Gamma} \ln G dw_\nu \quad (\nu = 1, 2, \dots, \rho) \quad (\text{see } (^{11,12})).$$

To study the solvability of the nonhomogeneous problem, let us consider the conjugate problem associated with it, consisting in finding on  $R - \Gamma$  a holomorphic differential  $d\Psi \in H_q$ , whose boundary values satisfy on  $\Gamma$  the condition

$$d\Psi^+ = \frac{1}{G} d\Psi^-. \quad (16)$$

Dividing (1) and multiplying (16) by  $X_0^+$ , we obtain two problems defined by the boundary conditions

$$\Phi_1^+ = G_0 \Phi_1^- + \frac{g}{X_0^+}, \quad (17)$$

$$d\Psi_1^+ = \frac{1}{G_0} d\Psi_1^-. \quad (18)$$

The solution of problem (17) is sought in the class  $H_p$ , and that of problem (18) in the class of differentials continuously extendable to  $\Gamma$ . Using assumption III on the functions  $G$  and  $g$ , one can show that problems (1) and (17) are equivalent and their solutions are related by the relation  $\Phi = X_0 \Phi_1$ . The equivalence of problems (16) and (18) and the validity of the relation  $d\Psi_1 = X_0 d\Psi$  are obvious.

**Theorem 3.** For problem (1) to be solvable, it is necessary and sufficient that

$$\int_{\Gamma} g d\Psi^+ = 0 \quad (19)$$

for every solution  $d\Psi$  of the conjugate problem.

**Necessity** follows from the Cauchy theorem.

**Sufficiency.** We rewrite condition (19) in the form

$$\int_{\Gamma} \frac{g}{X_0^+} d\Psi_1^+ = 0. \quad (20)$$

It is not difficult to show that on  $\Gamma$  one can construct a function  $g_1$ , satisfying the Hölder condition, such that the problem determined by the boundary condition

$$\Phi_2^+ = G_0 \Phi_2^- + g/X_0^+ + g_1,$$

will have a solution in the class  $H_p$ . From the necessity of condition (19) it follows that

$$\int_{\Gamma} \left( \frac{g}{X_0^+} + g_1 \right) d\Psi_1^+ = \int_{\Gamma} g_1 d\Psi_1^+ = 0. \quad (21)$$

But, as is known, from (21) there follows the existence of a function  $\Phi_3$ , holomorphic on  $R - \Gamma$  and continuously extendable to  $\Gamma$ , whose boundary values satisfy the condition

$$\Phi_3^+ = G_0 \Phi_3^- + g_1.$$

A direct verification shows that  $\Phi = \Phi_2 - \Phi_3$  is a solution of problem (17). Thus problem (17), and consequently problem (1), is solvable.

**Remark.** All those facts of the theory of boundary-value problems on Riemann surfaces with Hölder coefficients to which we have referred in the present note can be established starting from Theorem 1.

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*Note: Figure translations are in progress. See original paper for figures.*

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