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1967

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Abstract

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UDC 517.11

MATHEMATICS

FAN DINH ZIEU

ON SOME PROPERTIES OF CONSTRUCTIVE GENERALIZED FUNCTIONS

(Presented by Academician P. S. Novikov, October 22, 1966)

1. The present note is a continuation of the author's work ⁽⁵⁾. In this note all special terms and notations not explained are understood in the same way as in ⁽²⁻⁵⁾. In ⁽⁵⁾ the notion of a constructive generalized function was defined; it is a constructive analogue of the classical notion of a generalized function defined in ^(6, 7).

Let us introduce some further notions and notations: let Φ_1, Φ_2 be constructive generalized functions and let a, b be real duplexes, $a < b$. Take a fixed natural number k such that $k \geq \max(|a| \sqcup |b|)$. We shall say that Φ_1 is equal to Φ_2 in the interval $a \nabla b$, if there exists a word P of type $\rightarrow$ such that $L(P) < l^*$ and

$$\forall x (a < x < b \supset I(k \sqcup l - m \sqcup \{\Phi_1\}_k)(x) -$$

$$-I(k \sqcup l - n \sqcup \{\Phi_2\}_k)(x) = G(P)(x)),$$

where $m \Leftarrow |\Phi_1|_k$, $n \Leftarrow |\Phi_2|_k$, $l \Leftarrow \max(m \sqcup n)$, and G is the algorithm defined in Sec. 1 of ⁽⁵⁾.

It is not difficult to prove that in the indicated definition, instead of k one may take any natural number greater than or equal to $\max(|a| \sqcup |b|)$.

We shall say that Φ_1 is equal to Φ_2 in the interval $a \nabla \infty$, if there exists an algorithm $\mathfrak{P}$ of type $(\rightarrow)$ possessing the following properties:

- 1) $\forall k (k > a \supset \mathfrak{P}(k));$
- 2) $\forall k (k > a \supset L(\mathfrak{P}(k)) < l_k);$
- 3) $\forall k x (a < x \leq k \supset I(k \sqcup l_k - m_k \sqcup \{\Phi_1\}_k)(x) -$

$$-I(k \sqcup l_k - n_k \sqcup \{\Phi_2\}_k)(x) = G(\mathfrak{P}(k))(x)),$$

where $m_k \vartriangleleft |\Phi_1|_k$, $n_k \vartriangleleft |\Phi_2|_k$, $l_k \vartriangleleft \max(m_k \square n_k)$.

In an analogous way one may define the notion that Φ_1 is equal to Φ_2 in the interval $-\infty \nabla a$.

Theorem 1. *There exists an algorithm that transforms every word of the form $a \nabla b \square \Phi_1$, where a, b are real duplexes, $a < b$, and Φ_1 is a constructive generalized function, into a word of the form $n \square \mathbf{m}_1$ such that Φ_1 is equal to $\mathfrak{D}(n \square R(\mathbf{m}_1))$ in the interval $a \nabla b$.*

In other words, in every finite interval every constructive generalized function is equal to a derivative of some order of an almost uniformly continuous function.

* The notation $<$ is understood in the same way as in ⁽⁵⁾.

We shall say that the segment $a \triangle b$ is a **carrier** of the constructive generalized function Φ_1 , and shall write

$$(a \triangle b \text{ carries } \Phi_1),$$

if Φ_1 is equal to zero in the intervals $-\infty \nabla a$ and $b \nabla \infty$.

Let Φ_1 be a constructive generalized function. We shall say that Φ_1 is a **constructive generalized function with bounded carrier** if

$$\exists k (-k \triangle k \text{ carries } \Phi_1).$$

Theorem 2. *There is no algorithm that transforms each constructive generalized function Φ_1 with bounded carrier into a natural number k such that $(-k \triangle k \text{ carries } \Phi_1)$.*

Theorem 3. a) *Every constructive generalized function with bounded carrier is a constructive generalized function of finite order; b) there is no algorithm that transforms each constructive generalized function with bounded carrier Φ_1 into a natural number n such that Φ_1 is a constructive generalized function of order $\leq n$.*

2. Passing to the consideration of sequences of constructive generalized functions and of the limits of such sequences, we first introduce some notation: let $\mathbf{m}_1$ be a word of type $\mathbf{m}$, i.e. the complete code of an almost uniformly continuous function. We shall denote $\mathfrak{D}(n \square R(\mathbf{m}_1))$ by $\mathbf{m}_1^{(n)}$. If two constructive generalized functions Φ_1 and Φ_2 are equal in the interval $-k \nabla k$, we shall write $\Phi_1 \underset{k}{=} \Phi_2$.

Let λ_1 be an algorithm of type $(\mathbf{n} \rightarrow \mathbf{m})$ and X a word of type $\mathbf{m}$. We shall write $(X \text{ pred}_{C_k} \lambda_1)$ if $\xi(k \square X)$ is the limit of the sequence defined by the algorithm $(\tilde{\xi}_k \square \lambda_1)$ in the space C_k .

Let φ be an algorithm of type $(\mathbf{n} \rightarrow \Phi)$, i.e. a sequence of constructive generalized functions. We shall say that the constructive generalized function Φ_1 is the **limit of the sequence φ in the interval $-k\nabla k$** if

$$\exists i \mathbf{m}_1 \lambda_1 (\Phi_1 =_{\mathbf{k}} \mathbf{m}_1^{(i)} \ \& \ \forall_n (\varphi(n) =_{\mathbf{k}} \lambda_1(n)^{(i)}) \ \& \ (\mathbf{m}_1 \text{ pred}_{C_k} \lambda_1)),$$

where λ_1 is a variable for algorithms of type $(\mathbf{n} \rightarrow \mathbf{m})$.

We shall call Φ_1 the **limit of the sequence φ** , if for every k , Φ_1 is the limit of the sequence φ in the interval $-k\nabla k$.

Theorem 4. a) *Every constructive generalized function is the limit of a sequence of regular constructive generalized functions;* b) *every constructive generalized function is the limit of a sequence of constructive generalized functions with bounded carriers.*

It is not difficult to prove the following assertions:

- 1) If $\mathbf{m}_1$ is the limit in the space $\mathcal{C}$ of a sequence λ of words of type $\mathbf{m}$ (i.e. for every k , $\xi(k \square \mathbf{m}_1)$ is the limit in C_k of the sequence defined by the algorithm $(\tilde{\xi}_k \square \circ \lambda)$), then $R(\mathbf{m}_1)$ is the limit of a sequence of constructive generalized functions defined by the algorithm $(R \circ \lambda)$.
- 2) If Φ_1 is the limit of the sequence φ of constructive generalized functions, then for whatever natural number m , $\mathfrak{D}(m \square \Phi_1)$ is the limit of the sequence of constructive generalized functions defined by the algorithm $(\mathfrak{D}_m \square \circ \varphi)$.

Let a real duplex α_0 be given. Let k be a natural number, $\mathbf{m}_1^k$ a word of type $\mathbf{m}^k$, and λ an algorithm of type $(\mathfrak{d} \rightarrow \mathbf{m}^k)$, applicable to any-

to any duplex $\alpha \neq \alpha_0$. We shall say that $\mathbf{m}_1^k$ is the **limit** of λ as $\alpha \rightarrow \alpha_0$ in the space C_k , if

$$\forall i \exists j \forall \alpha (-k \leq x \leq k \ \& \ 0 < |\alpha - \alpha_0| < 2^{-j} \supset |\mathbf{m}_1^k(x) - \lambda(\alpha)(x)| < 2^{-i}).$$

Let $\mathbf{m}_1$ be a word of type $\mathbf{m}$, and let λ be an algorithm of type $(\mathfrak{d} \rightarrow \mathbf{m})$, applicable to every duplex $\alpha \neq \alpha_0$. We shall say that $\mathbf{m}_1$ is the **limit** of λ as $\alpha \rightarrow \alpha_0$ in the space C_k , and shall write

$$(\mathbf{m}_1 \text{ pred}_{C_k(\alpha \rightarrow \alpha_0)} \lambda),$$

if $\xi(k \square \mathbf{m}_1)$ is the limit of $(\tilde{\xi}_k \square \circ \lambda)$ as $\alpha \rightarrow \alpha_0$ in the space C_k .

Now let φ be an algorithm of type $(\mathfrak{d} \rightarrow \Phi)$, applicable to every duplex $\alpha \neq \alpha_0$. We shall say that the constructive generalized function Φ_1 is the **limit** of φ as $\alpha \rightarrow \alpha_0$, if for every natural number k the following holds:

$$\exists i j \mathbf{m}_1 \lambda_1 (\Phi_1 = \mathbf{m}_1^{(i)} \& \forall \alpha (0 < |\alpha - \alpha_0| < 2^{-j} \supset$$

$$\supset \varphi(\alpha) = \lambda_1(\alpha)^{(i)} \& (\mathbf{m}_1 \text{ pred}_{C_k(\alpha \rightarrow \alpha_0)} \lambda_1)),$$

where λ_1 is a variable for algorithms of type $(\mathfrak{d} \rightarrow \mathfrak{m})$, applicable to any duplex $\alpha \neq \alpha_0$.

3. We define the operator of a linear change of variable for constructive generalized functions.

Let $\alpha \neq 0$ and β be real duplexes, and let $\Phi_1 \simeq X\tau Y\tau Z\tau T$ be a constructive generalized function. Construct an algorithm η of type $(\mathfrak{n} \rightarrow \mathfrak{n})$ such that

$$\forall k (\eta(k) > |\alpha|k + |\beta|).$$

Construct algorithms λ of type $(\mathfrak{n} \rightarrow \mathfrak{m})$, N of type $(\mathfrak{n} \rightarrow \mathfrak{n})$, Q of type $(\mathfrak{nn} \rightarrow \mathfrak{n})$, such that

$$\forall n x (\lambda(n)(x) \simeq \langle X \rangle(n)(\alpha x + \beta));$$

$$\forall k (N(k) \simeq \langle Y \rangle(\eta(k)));$$

$$\forall k m (Q(k \square m) \simeq \langle T \rangle(\eta(k) \square m)).$$

It is possible to construct an algorithm P of type $(\mathfrak{nn} \rightarrow \mathfrak{n})$ such that, for any k and n , $L(P(k \square n)) < p$ ($p \simeq N(k)$), and for every x in $-k \triangle k$:

$$\mathfrak{S}(p \square \lambda(n))(x) + G(P(k \square n))(x) =$$

$$= \frac{1}{\alpha^p} \cdot \mathfrak{S}(p \square \langle X \rangle(n)) + G(\langle Z \rangle(\eta(k) \square n))(\alpha x + \beta).$$

It is not difficult to prove that the word $\xi \lambda 3 \tau \xi N 3 \tau \xi P 3 \tau \xi Q 3$ represents a constructive generalized function. One can construct an algorithm which transforms each word of the form $\alpha \square \beta \square \Phi_1$ into the corresponding word $\xi \lambda 3 \tau \xi N 3 \tau \xi P 3 \tau \xi Q 3$. We shall denote this algorithm by $\mathcal{L}$.

Let c be a real duplex. Denote by $\tilde{c}$ a word of type $\mathfrak{m}$ such that

$$\forall x (\tilde{c}(x) = c),$$

and then denote $R(\tilde{c})$ by $\bar{c}$. Then $\bar{c}$ is a constructive generalized function.

A constructive generalized function Φ_1 is called **constant** if there exists a real duplex c such that $\Phi_1 = \bar{c}$.

Let a constructive generalized function Φ_1 and a real duplex x_0 be given. We construct an algorithm φ of type $(D \rightarrow \Phi)$, such that for any $a \neq 0$:

$$\varphi(a) \simeq \mathcal{L}(a \square x_0 \square \Phi_1). \quad (1)$$

Copying the classical proof (see ⁽⁶⁾, § 16), we can prove the following assertion: if there exists such a constructive generalized function Φ_2 that Φ_2 is the limit of φ as $a \rightarrow 0$, then Φ_2 is constant.

We shall say that a real duplex c is a **value** of the constructive generalized function Φ_1 at the point x_0 , and shall write

$$(c \underline{\text{val}}_{x_0} \Phi_1),$$

if $\bar{c}$ is the limit of the algorithm φ , defined by formula (1), as $a \rightarrow 0$.

The point x_0 is called a **regular point** of the constructive generalized function Φ_1 if $\exists c(c \underline{\text{val}}_{x_0} \Phi_1)$; it is called a **singular point** of Φ_1 if it is not a regular point of Φ_1 .

It is easy to prove that if Φ_1 is equal to $R(\mathfrak{sh}_1)$ in some interval $a \nabla b$, where $\mathfrak{sh}_1$ is a word of type $\mathfrak{sh}$, then every point x_0 in $a \nabla b$ is a regular point of Φ_1 , and whatever the point x_0 in $a \nabla b$, $(\mathfrak{sh}_1(x_0) \underline{\text{val}}_{x_0} \Phi_1)$.

Łojasiewicz's theorem (Theorem 2.3 ⁽⁸⁾) is transferred into constructive mathematics in the following way.

Let Φ_1 be a constructive generalized function and x_0 a real duplex. Let $k > |x_0|$. The real duplex c is a value of Φ_1 at the point x_0 if and only if there exist a number n and a word $\mathfrak{sh}_1$ such that Φ_1 is equal to $\mathfrak{sh}_1^{(n)}$ in $-k \nabla k$, and $c/n!$ is the limit of the function f , defined by the formula

$$\forall x (f(x) \simeq \mathfrak{sh}_1(x) : (x - x_0)^n),$$

as $x \rightarrow x_0$.

Theorem 5. *One can construct a constructive generalized function Φ_1 possessing the property: there is no function f of a real variable, defined at all regular points of Φ_1 , and such that at every regular point x_0 of Φ_1 one has $(f(x_0) \underline{\text{val}}_{x_0} \Phi_1)$.*

The proof of this theorem is based on Theorem 5.1 from ⁽¹⁾ and the lemma of § 1, Chapter III from ⁽⁴⁾.

Theorem 6. *There exists such a constructive everywhere-defined function f of a real variable that there is no constructive generalized function Φ_1 satisfying the condition:*

$$\forall x (f(x) \underline{\text{val}}_x \Phi_1).$$

The question of a constructive analogue of the concept of generalized function, based on the concept of a linear continuous functional in the space of finite infinitely differentiable functions, will be considered in another communication.

The author expresses deep gratitude to A. A. Markov and N. A. Shanin for valuable advice.

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Received
15 X 1966

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Note: Figure translations are in progress. See original paper for figures.

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