

EXISTENCE AND UNIQUENESS THEOREMS FOR SOLUTIONS OF HAMMERSTEIN EQUATIONS

P. P. Zabreiko, A. I. Povolotskii

1967

SovietRxiv

View the original and related papers at <https://sovietsrxiv.org/items/ru-196701.59743>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.948.33

MATHEMATICS

P. P. ZABREIKO, A. I. POVOLOTSKII

EXISTENCE AND UNIQUENESS THEOREMS FOR SOLUTIONS OF HAMMERSTEIN EQUATIONS

(Presented by Academician V. I. Smirnov, 19 XII 1966)

The paper considers the nonlinear Hammerstein integral equation

$$x(t) = \int_{\Omega} k(t, s) f[s, x(s)] ds + h(t). \quad (1)$$

Here Ω is a bounded closed set of a finite-dimensional space; $f(s, u)$ is an operator satisfying the Carathéodory conditions and acting from $\Omega \times R^n$ into R^n ; $k(t, s)$ ($t, s \in \Omega$) is a symmetric (i.e., $k(s, t) = [k(t, s)]^*$) matrix, measurable in the aggregate of the variables; R^n is the real n -dimensional space. Such equations have been studied by methods of functional analysis in ⁽¹⁻⁸⁾; the principal constructions in this connection were carried out in spaces of vector-functions C , $\mathcal{L}_p$, and in Orlicz spaces. In the present note the investigation of equations (1) is carried out in general functional spaces, whose theory is set forth, for example, in ⁽⁹⁾. This approach makes it possible to formulate several simple assertions on the solvability of equation (1), which, when applied to concrete spaces, contain a large part of the previously known results. The transition to general spaces has already made it possible, for scalar equations, to prove finer results, new even for the spaces $\mathcal{L}_p$.

The theorems presented in the present article reduce the investigation of Hammerstein equations to the study, in various spaces, of the linear integral operator K with matrix-kernel $k(t, s)$ and of the superposition operator $fx(s) = f[s, x(s)]$. Various theorems in this direction are set forth, for example, in ⁽⁹⁻¹¹⁾.

Let us also note that the results of the article are naturally carried over to equations with a Lebesgue integral with respect to an arbitrary measure, in particular, to infinite systems; in this case, instead of Banach spaces, one may consider locally convex spaces.

1. Denote by S the space of measurable almost everywhere finite vector-functions on Ω with values in R^n . Put

$$\langle x, y \rangle = \int_{\Omega} (x(s), y(s)) ds. \quad (2)$$

A Banach space E of vector-functions from S is called **ideal** if from $|x| \leq |y|$, $x \in S$, $y \in E$, it follows that $x \in E$ and $\|x\|_E \leq \|y\|_E$ (by $|x|$ is denoted the vector whose components are equal to the moduli of the components x ; inequalities for vectors are understood componentwise). Denote by E^0 the totality of elements of E with absolutely continuous norm.

Every ideal space of functions with values in R^n may be regarded as the direct sum of n spaces $E_1, \dots, E_n$ of scalar functions. Let Ω_i ($i = 1, \dots, n$) be the supports of E_i , i.e., such subsets of Ω that every function from E_i vanishes outside Ω_i , while in E_i there exist functions positive for $s \in \Omega_i$.

We shall call ideal spaces E and F **dual** if

$$\|x\|_E = \sup_{\|y\|_F \leq 1} \langle x, y \rangle, \quad \|y\|_F = \sup_{\|x\|_E \leq 1} \langle x, y \rangle. \quad (3)$$

For each ideal space E , by E' we shall denote the space of vector-functions whose components vanish outside the supports Ω_i of the spaces E_i , and for which the norm

$$\|y\|_{E'} = \sup_{\|x\|_E \leq 1} \langle x, y \rangle \quad (4)$$

is meaningful.

The spaces E and E' are dual if and only if, for any sequence $x_m \in E$ converging in measure to $x \in E$, the inequality

$$\|x\|_E \leq \lim_{m \rightarrow \infty} \|x_m\|_E$$

holds.

Among ideal spaces are the space E_{u_0} (u_0 is a nonnegative function from S) of vector-functions for which the norm

$$\|x\|_{E_{u_0}} = \inf\{\lambda : |x| \leq \lambda u_0\}, \quad (5)$$

is meaningful; the space E'_{u_0} dual to it; the space $\mathcal{L}_p$; the Orlicz spaces, and many others.

Let $E_1, \dots, E_m, X$ be ideal spaces of scalar functions; $k_1, \dots, k_m$ nonnegative numbers. We shall write $X < (E_1^{k_1}, \dots, E_m^{k_m})$ if

$$\| |x_1|^{k_1} \dots |x_m|^{k_m} \|_X \leq \|x_1\|_{E_1}^{k_1} \dots \|x_m\|_{E_m}^{k_m} \quad (x_1 \in E_1, \dots, x_m \in E_m). \quad (6)$$

Below it is assumed that E and F are such dual spaces that $E \subseteq \mathcal{L}_2$, the superposition operator f acts from E to F , and the linear operator K acts from F to E .

2. An operator Q acting from one space into another will be called **asymptotically quadratic** if, for some quadratic operator Q_∞ ,

$$\lim_{\|x\| \rightarrow \infty} \frac{\|Qx - Q_\infty x\|}{\|x\|^2} = 0. \quad (7)$$

An important role in what follows is played by the asymptotic quadraticity of the superposition operator, acting from some ideal space $E \subseteq \mathcal{L}_2$ of functions with values in R^n into the space $\mathcal{L}_1$ of scalar functions,

$$Qx(s) = Q[s, x(s)], \quad (8)$$

where $Q(s, u)$ is some function satisfying the Carathéodory conditions. It can be shown that from the asymptotic quadraticity of Q it follows that the function $Q(s, u)$ admits the representation

$$Q(s, u) = \sum_{i,j=1}^n q_{ij}(s) u_i u_j + \omega(s, u), \quad (9)$$

where the function $\omega(s, u)/(u, u)$ tends in measure to zero as $u \rightarrow \infty$, and

$$Q_\infty x(s) = \sum_{i,j=1}^n q_{ij}(s) x_i(s) x_j(s) \quad (x = \{x_1, \dots, x_n\} \in E). \quad (10)$$

Lemma 1. Suppose the function $Q(s, u)$ admits the representation (9), where $q_{ij} \in E'_{ij}$ and $E_{ij} < (E_i, E_j)$. Suppose, moreover, that

$$\lim_{r \rightarrow \infty, \text{mes } D \rightarrow 0} \frac{1}{r^2} \sup_{\|x\|_E = r} \int_D |Q[s, x(s)]| ds = 0. \quad (11)$$

Then Q is asymptotically quadratic.

A sufficient condition for (11) to be satisfied is the inequality

$$|Q(s, u)| \leq \sum_{i_1, \dots, i_n} a_{i_1 \dots i_n}(s) |u_1|^{k_1} \dots |u_n|^{k_n}, \quad (12)$$

where $k_1 + \dots + k_n \leq 2$, $a_{i_1 \dots i_n} \in (E'_{i_1 \dots i_n})^0$, and $E_{k_1 \dots k_n} < (E_1^{k_1}, \dots, E_n^{k_n})$.

Below, $Q(s, u)$ will always denote a function determining a bounded asymptotically quadratic operator Q acting from E into $\mathcal{L}_1$, $Q_\infty = 0$.

Let us introduce one more notation. Let $q(s) = (q_{ij}(s))$ be some symmetric matrix ($[q(s)]^* = q(s)$), and let $\bar{E} = \mathcal{L}_2$. Put

$$m(q; \bar{E}) = \sup_{x \neq 0} \langle qx, x \rangle / \|x\|_{\bar{E}}^2. \quad (13)$$

In practically important cases this quantity is easily computed or estimated.

3. Suppose that K is positive definite as an operator in $\mathcal{L}_2$. Denote by H the set of values of the operator $K^{1/2}$ on $\mathcal{L}_2$; it is well known that $H \subseteq E$. Below, $\bar{E}$ is some ideal space for which $E \subseteq \bar{E} \subseteq \mathcal{L}_2$. Using M. A. Krasnosel'skiĭ's fixed-point principle^(5,7), we obtain:

Theorem 1. *Let the operator $K^{1/2}fK^{1/2}$ be completely continuous in $\mathcal{L}_2$, and let $h \in H$. Suppose that the inequality*

$$(u, f(s, u)) \leq \sum_{i,j=1}^n q_{ij}(s) u_i u_j + Q(s, u), \quad (14)$$

holds, where $q(s)$ is a symmetric matrix, and moreover $m(q; \bar{E}) \|K\|_{\bar{E}' \rightarrow \bar{E}} < 1$.

Then equation (1) has a solution in H .

Suppose now that equation (1) is potential, i.e., that there exists a scalar function $\Phi(s, u)$ ($\Phi(s, 0) = 0$) such that $\text{grad } \Phi(s, u) = f(s, u)$. Then consider the functional defined on $\mathcal{L}_2$

$$\Phi(x) = \int_{\Omega} \Phi[s, K^{1/2}x(s)] ds. \quad (15)$$

Theorem 2. *Let the functional Φ be weakly lower semicontinuous, and let $h \in H$. Suppose that the inequality*

$$\Phi(s, u) \leq \frac{1}{2} \sum_{i,j=1}^n q_{ij}(s) u_i u_j + Q(s, u), \quad (16)$$

holds, where $q(s)$ is a symmetric matrix, and moreover $m(q; \bar{E}) \|K\|_{\bar{E}' \rightarrow \bar{E}} < 1$.

Then equation (1) has a solution in H .

We also give a uniqueness theorem.

Theorem 3. *Suppose that the inequality*

$$(u - v, f(s, u) - f(s, v)) < \sum_{i,j=1}^n q_{ij}(s)(u_i - v_i)(u_j - v_j) \quad (u \neq v), \quad (17)$$

holds, where $q(s)$ is a symmetric matrix, and moreover $m(q; \bar{E})\|K\|_{\bar{E}' \rightarrow E} \leq 1$.

Then equation (1) has at most one solution.

4. Suppose that the operator K has a finite number of negative eigenvalues, and that $(-\lambda_0)$ is the largest of them. Denote by $\tilde{H}$ the set of values on $\mathcal{L}_2$ of the operator $\tilde{K}^{1/2}$, where $\tilde{K}$ is the positive definite self-adjoint quadratic root of the operator K^2 . Note that the operator $\tilde{K}^{1/2}f\tilde{K}^{1/2}$ acts in $\mathcal{L}_2$, and, in the case where equation (1) is potential, the functional

$$\tilde{\Phi}(x) = \int_{\Omega} \Phi[s, \tilde{K}^{1/2}x(s)] ds. \quad (18)$$

Theorem 4. Let the operator $\tilde{K}^{1/2}f\tilde{K}^{1/2}$ be completely continuous and let $h \in \tilde{H}$. Suppose the inequality

$$(u, f(s, u)) \leq \sum_{i,j=1}^n q_{ij}u_iu_j + Q(s, u), \quad (19)$$

holds, where q is a symmetric matrix, and $m(q; \mathcal{L}_2)\lambda_0 < -1$.

Then equation (1) has a solution in $\tilde{H}$.

Theorem 5. Let the functional Φ be weakly upper semicontinuous and let $h \in \tilde{H}$. Suppose the inequality

$$\Phi(s, u) \leq \frac{1}{2} \sum_{i,j=1}^n q_{ij}u_iu_j + Q(s, u), \quad (20)$$

holds, where q is a symmetric matrix and $m(q; \mathcal{L}_2)\lambda_0 < -1$.

Then equation (1) has a solution in $\tilde{H}$.

Theorem 6. Suppose the inequality

$$(u - v, f(s, u) - f(s, v)) < \sum_{i,j=1}^n q_{ij}(u_i - v_i)(u_j - v_j) \quad (u \neq v), \quad (21)$$

holds, where q is a symmetric matrix, with $m(q; \mathcal{L}_2)\lambda_0 \leq -1$.

Then equation (1) has at most one solution.

5. Below, $C = K^{1/2}fK^{1/2}$ and $\Psi = \Phi$ in the case of positive definite K , and $C = \tilde{K}^{1/2}f\tilde{K}^{1/2}$ and $\Psi = \tilde{\Phi}$ in the case when K has a finite number of negative eigenvalues. We give sufficient conditions for the complete continuity of C and the weak upper semicontinuity of Ψ .

Lemma 2. Suppose one of the following conditions is satisfied:

- a) K is completely continuous as an operator from F into E ;
- b) K is regular as an operator from F into E , and f is an improving ⁽¹⁰⁾ operator from E into F ;
- c) $E = E_{u_0}$, $F = E'_{u_0}$.

Then C is completely continuous in $\mathcal{L}_2$, and, in the case when equation (1) is potential, Ψ is weakly upper semicontinuous on $\mathcal{L}_2$.

Lemma 3. Let equation (1) be potential, and suppose that the function $\Phi(s, u)$ satisfies the inequality

$$\Phi(s, u + h) - \Phi(s, u) - (f(s, u), h) \leq \mathcal{L}(s, u, h), \quad (22)$$

where $\mathcal{L}(s, u, h)$ ($\mathcal{L}(s, u, 0) \equiv 0$) is a function satisfying the Carathéodory conditions and defining a superposition operator $L(x, h)$ acting from $E \times E$ into $\mathcal{L}_1$, improving for each fixed $x \in E$.

Then Ψ is weakly upper semicontinuous in $\mathcal{L}_2$.

The authors express their gratitude to their supervisor M. A. Krasnosel'skii.

Leningrad State
Pedagogical Institute
named after A. I. Herzen

Received
14 XII 1966

REFERENCES

1. M. M. Vainberg, *Variational Methods for the Study of Nonlinear Operators*, 1956.
2. M. M. Vainberg, I. V. Shragin, *Izv. Vyssh. Uchebn. Zaved.*, Mathematics, No. 1 (44), 17 (1965).
3. A. Hammerstein, *Acta Math.*, 54, 117 (1930).
4. M. Holom, *Publ. Math. Univ. Belgrade*, 5, 52 (1936).

5. M. A. Krasnosel' skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, 1956.
6. M. A. Krasnosel' skii, Ya. B. Rutitskii, *Tr. Mosk. Mat. Obshch.*, 7, 63 (1958).
7. A. I. Povolotskii, *DAN*, 99, No. 6, 901 (1954).
8. *Nonlinear Integral Equations*, Madison, 1964.
9. P. P. Zabreiko, *Tr. Seminar on Functional Analysis*, issue 3 (1966).
10. M. A. Krasnosel' skii, P. P. Zabreiko et al., *Integral Operators in Spaces of Summable Functions*, Moscow, 1966.
11. M. A. Krasnosel' skii, Ya. B. Rutitskii, *Convex Functions and Orlicz Spaces*, Moscow, 1958.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.