

TOPOLOGICAL INVARIANTS OF NONCOMPACT LIE GROUPS RELATED TO INFINITE- DIMENSIONAL REPRESENTATIONS

I. M. Gel'fand, D. B. Fuchs

1967

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196701.59330>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 513.836+519.46

MATHEMATICS

Corresponding Member of the Academy of Sciences of the USSR I. M. GELFAND, D. B. FUKS

TOPOLOGICAL INVARIANTS OF NONCOMPACT LIE GROUPS RELATED TO INFINITE-DIMENSIONAL REPRESENTATIONS

1. It is known that the algebraic-topological invariants of a compact Lie group G and of its classifying space B_G are closely connected with representations of the group G . For example, to each unitary or orthogonal representation of the group G one can construct a vector U - (or O -) bundle with base B_G . As Atiyah showed, this construction gives an isomorphism between $KU(B_G)$ (or $KO(B_G)$) and the completed ring of unitary (orthogonal) representations of the group G . If, however, the group G is noncompact, then all its homotopy invariants are the same as those of its maximal compact subgroup $\hat{G}$; as for the classifying spaces, one has the equality $B_G = B_{\hat{G}}$. The representations of the groups G and $\hat{G}$, however, have nothing in common.

As is known, a substantive theory of representations of noncompact Lie groups arises only upon passing to infinite-dimensional representations. Therefore a generalization of the results of Atiyah and Peterson to the case of noncompact groups must in one way or another be connected with vector bundles with infinite-dimensional fiber. Such bundles are considered in the works of Dixmier and Douady^(1,2). The basic idea expressed in these works is that no substantive analogue of K -theory can be constructed by means of infinite-dimensional vector bundles.

The consideration of classifying spaces B_G of a Lie group in the usual sense is equivalent to considering the category of principal G -bundles whose bases are CW -complexes. The coincidence of the spaces B_G and $B_{\hat{G}}$ means the coincidence of the categories of principal G - and $\hat{G}$ -bundles with CW -complex bases.

In our preceding note⁽³⁾ we considered a broader category of $\hat{G}$ -bundles, of whose bases we require only Hausdorffness. In it one computes the group of characteristic classes of such bundles with values in cohomology with coefficients in the sheaf of germs of continuous functions on the base. These groups are expressed in terms of well-studied invariants of Lie groups; moreover, they turn out to be nontrivial only for noncompact groups. The nontriviality of these

groups indicates the essential nature of the enlargement of the category, since the cohomology of any CW -complex (and, in general, of any paracompact space) with coefficients in the sheaf of germs of continuous functions is equal to zero.

In the present note we establish certain relations between similar invariants and representations (in particular, infinite-dimensional ones) of noncompact Lie groups. In particular, every infinite-dimensional representation of the group G in a certain sense induces an infinite-dimensional vector bundle over a certain nonparacompact space, which replaces for us the classifying space.

2. We define the category of principal G -bundles in the same way as in ⁽³⁾. We regard the group G as a closed Lie subgroup of the group $GL(n, \mathbf{R})$. By $\mathcal{E}^I$, where I is some set, we denote the space of all n -frames in the Tikhonov product

$$\mathcal{E}^I = \prod_{i \in I} \mathbf{R}_i$$

of lines indexed by the elements of I . On the space $\mathcal{E}^I$ the group $GL(n, \mathbf{R})$ and its closed subgroup G act without fixed points. The orbit space

$\mathcal{E}^I/G = S_G^I$ is a Hausdorff, but not regular (if the group G is not compact) space. The bundle $p^I : \mathcal{E}^I \rightarrow S_G^I$ is universal in the sense that for any G -bundle $p : E \rightarrow X$ there are a set I and a map

$$\begin{array}{ccc} E & \xrightarrow{\varphi''} & \mathcal{E}^I \\ p \downarrow & & \downarrow p^I \\ X & \xrightarrow{\varphi'} & S_G^I \end{array}$$

of bundles.

Let H be any topological group. Denote by $EH(X)$ the set of principal H -bundles with base X . To a map $f : X \rightarrow Y$ there corresponds a map $f^* : EH(Y) \rightarrow EH(X)$. We define the set $EH_{alg}(G)$ as follows. An element $a \in EH_{alg}(G)$ is a function assigning to each principal G -bundle $(\xi), p : E \rightarrow X$, a principal H -bundle $a(\xi) \in EH(X)$, with the property that if

$$\begin{array}{ccc} E_1 & \xrightarrow{\varphi''} & E_2 \\ p_1 \downarrow & & \downarrow p_2 \\ X_1 & \xrightarrow{\varphi'} & X_2 \end{array}$$

is a map of the bundle $(\xi_1), p_1 : E_1 \rightarrow X_1$, into the bundle $(\xi_2), p_2 : E_2 \rightarrow X_2$, then $(\varphi')^* a(\xi_2) = a(\xi_1)$.

Clearly, if we restricted ourselves to considering G -bundles over CW -complexes, then the analogously defined set $EH_{alg}(G)$ would naturally be, for G and H , equivalent to $EH(B_G) = EH(BG)$.

If a topological homomorphism $\eta : G \rightarrow H$ is given, then to any G -bundle (ξ) , $p : E \rightarrow X$, one can associate the principal H -bundle $(\eta^*\xi)$, $p' : E \times H/G \rightarrow X$, where the group G acts on $E \times H$ by the formula $(x, h)g = (xg, \eta(g^{-1})h)$. The collection of elements $\eta^*\xi \in EH(x)$ determines an element $\eta^* \in EH_{alg}(G)$.

Theorem 1. *For any element $a \in EH_{alg}(G)$ there exists a homomorphism $\eta : G \rightarrow H$ such that $a = \eta^*$. Moreover, $\eta_1^* = \eta_2^*$ if and only if there exists $h \in H$ such that $\eta_1(g) = h\eta_2(g)h^{-1}$ for all $g \in G$.*

The plan of the proof is as follows. Let (ξ) , $p : E \rightarrow X$, be a principal G -bundle. The principal G -bundle $p^*\xi$, induced by the bundle ξ under the map p , is trivial. Therefore the H -bundle $p^*a(\xi) \in EH(E)$ is also trivial. The fibers of this bundle over the points $x, xg \in E$ are canonically isomorphic to the fiber of the bundle $a(\xi)$ over the point $p(x) = p(xg) \in X$. Fix an element $g \in G$ and an H -isomorphism between the H -bundle $p^*a(\xi)$ and the trivial bundle $E \times H \rightarrow E$, that is, assign to each point of the space of the bundle $p^*(a(\xi))$ a pair (x, h) , where $x \in E$, $h \in H$, and the action of the group H on this space is given by the formula $(x, h)h_0 = (x, hh_0)$. Let the canonical isomorphism of the fibers over the points x and xg , for any $x \in E$, carry the point $(x, 1)$ to the point $(xg, \psi_g(x))$, where 1 is the identity of the group H , $\psi_g(x) \in H$. We obtain a continuous map $\psi_g : E \rightarrow H$, which is constructed from the bundle $a(\xi)$ non-uniquely, but characterizes it completely.

Next it is proved that the map ψ_g can be made constant, i.e. $\psi_g(x) = \eta(g) \in H$ for any $x \in E$, where $\eta : G \rightarrow H$ is a topological homomorphism independent of the bundle ξ . Here one uses the functorial dependence of $a(\xi)$ on ξ , the universality of the bundles $p^I : \mathcal{E}^I \rightarrow S_G^I$, and the following lemma.

Lemma. *Let Y be any space, and let $f : \mathcal{E} \rightarrow Y$ be any continuous map. There exists $I' \subset I$, where the cardinality of the set $I \setminus I'$ does not exceed the cardinality of a base of neighborhoods of the space Y , such that the map f is constant on $\mathcal{E}^{I'} \subset \mathcal{E}^I$.*

The universality of the bundles ξ^I makes it possible to define $EH_{alg}(G)$ starting not from all G -bundles, but only from a single G -bundle ξ^I , where

I is any set whose cardinality is strictly greater than the cardinality of a base of neighborhoods of the group G . In other words, $EH_{alg}(G)$ is the subset of the set $EH(S_G^I)$ consisting of elements that are mapped into themselves under all mappings $S_G^I \rightarrow S_G^I$ included in a diagram of the form

$$\begin{array}{ccc} \mathcal{E}^I & \longrightarrow & \mathcal{E}^I \\ \downarrow p^I & & \downarrow p^I \\ S_G^I & \longrightarrow & S_G^I. \end{array}$$

If H is the group of unitary transformations of a Hilbert space V , then the set $EH(Y)$, where Y is any space, can be interpreted as the set of all vector bundles over Y with fiber V . Thus the following holds.

Corollary. *The set of all equivalence classes of representations of the group G in the Hilbert space V is in a natural one-to-one correspondence with the set of equivalence classes of vector bundles over S_G^I , where I has the cardinality of the continuum, invariant with respect to the mappings $S_G^I \rightarrow S_G^I$ described above.*

Examples. 1°. Let $G = \mathbf{R}$. Then a point of $\mathcal{E}^I$ is a set $\{x_i, i \in I\}$ of real numbers, not all of which are zero; $S_G^I = S^I$ is obtained from $\mathcal{E}^I$ by identifying $\{x_i\} = \{x_i\alpha\}$, where $\alpha > 0$. Consider a representation of the group G in the space $L_2(-\infty, +\infty)$ of square-integrable functions on the whole line. It corresponds to a bundle $E \rightarrow S^I$, in which a point of E is a function of class L_2 defined on the complete inverse image of one of the points of the space S^I under the mapping $\mathcal{E}^I \rightarrow S^I$; more precisely, a point of E is a function $f(\{x_i, i \in I\})$, defined if $x_i = x_i^0\alpha$, $i \in I$, where the x_i^0 are fixed and $\alpha > 0$, and such that the function F given on the line by the formula $F(t) = f(\{x_i^0 t\})$ belongs to $L_2(-\infty, +\infty)$.

2°. Let $G = SL(2, C)$, the Lorentz group. The space S_G^I in this case is the set of two-dimensional subspaces of the space $\mathcal{E}^I$, in each of which a frame is specified up to a unimodular transformation. Now consider the following bundle over the space S_G^I . A point of the space of this bundle is a collection consisting of a plane x with the structure indicated in it and a function $f(z) = z = (z_1, z_2) \in x$, defined in this plane, satisfying the functional equation

$$f(\lambda z) = \lambda^{n_1-1} \bar{\lambda}^{n_2-1} f(z),$$

where $n_1 - n_2 = n$ is an integer (a homogeneous function of z and $\bar{z}$ of dimensions $n_1 - 1, n_2 - 1$). These functions satisfy the condition of square integrability (for details see [5]) and natural conditions of continuity with respect to the plane. The vector bundles constructed are nontrivial and distinct. Such bundles are in one-to-one correspondence with the unitary representations of the so-called principal series of the Lorentz group.

3. Let M be any (in particular, infinite-dimensional) representation of the group G . We shall say that an element $a \in H_{alg}^q(G; M)$ is given if to every principal G -bundle $(\xi), p: E \rightarrow X$ (with Hausdorff base), there is assigned an element $a(\xi) \in H^q(X; M)$, and if

$$\begin{array}{ccc} E_1 & \xrightarrow{\varphi''} & E_2 \\ \downarrow p_1 & & \downarrow p_2 \\ X_1 & \xrightarrow{\varphi'} & X_2 \end{array}$$

is a mapping of the bundle $(\xi_1), p_1 : E_1 \rightarrow X_1$, into the bundle $(\xi_2), p_2 : E_2 \rightarrow X_2$, then $(\varphi')^*a(\xi_2) = a(\xi_1)$. Here $H^q(X; M)$ is the cohomology group of the space X with coefficients in the sheaf of germs of continuous sections of the vector bundle $p^\# : E \times M/G \rightarrow X$ with fiber M .

Consider the homogeneous space $G^* = G/\hat{G}$ and the space $\Omega^q(G^*; M)$ of smooth differential forms of degree q with coefficients in the space M . The action of the group G on $\Omega^q(G^*; M)$ is defined by the formula

$$(g\varphi)(l_1, \dots, l_q) = g(\varphi(g^{-1}l_1, \dots, g^{-1}l_q)),$$

where $l_1, \dots, l_q$ are tangent vectors to the manifold G^* at some point $x \in G^*$; $g^{-1}l_1, \dots, g^{-1}l_q$ are their images under the action of the element $g^{-1} \in G$. By $\Omega_S^q(G^*; M)$ we shall denote the totality of invariant forms, i.e., such forms $\varphi \in \Omega^q(G^*; M)$ that $g\varphi = \varphi$ for all $g \in G$.

Theorem 2. The group $H_{alg}^q(G; M)$ is isomorphic to the cohomology group of the complex

$$\Omega_S^0(G^*; M) \xrightarrow{d_1} \Omega_S^1(G^*; M) \xrightarrow{d_2} \dots,$$

where d_i are the differentials.

Theorem 3. $H_{alg}^q(G; M) = \text{Ext}^q(1; M)$.

Here Ext is considered in the category of all representations of the group G ; by a representation we mean an action of the group G on a linear topological space M as a group of linear operators such that the mapping $G \times M \rightarrow M$ induced by this action is continuous.

As is known ⁽⁴⁾, the group $\text{Ext}^q(1, M)$ can be defined as the set of exact sequences

$$0 \rightarrow 1 \rightarrow A_1 \rightarrow \dots \rightarrow A_q \rightarrow M \rightarrow 0$$

with a certain equivalence relation. Therefore the elements of $H_{alg}^q(G; M)$ correspond to such exact sequences.

For example, $H_{alg}^q(\mathbf{R}; 1)$ is equal to $\mathbf{R}$ for $q = 0, 1$ and is equal to 0 for $q > 1$, which corresponds to the fact that all exact sequences of representations of the group $\mathbf{R}$ of the form

$$0 \rightarrow 1 \rightarrow A_1 \rightarrow \dots \rightarrow A_q \rightarrow 1 \rightarrow 0$$

are reduced by the equivalence relation mentioned above to the following two:

$$0 \rightarrow 1 \rightarrow 1 \rightarrow 0$$

and

$$0 \rightarrow 1 \rightarrow J \rightarrow 1 \rightarrow 0,$$

where J is the Jordan representation of the group $\mathbf{R}$ in the plane, i.e.,

$$J_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad \text{for } a \in \mathbf{R}.$$

If the group G is semisimple, then every finite-dimensional representation is completely reducible, and therefore the results obtained essentially pertain to infinite-dimensional representations.

For example, if $G = SL(2; \mathbf{C})$ is the Lorentz group, then

$$H_{alg}^q(SL(2, \mathbf{C}), 1) = \begin{cases} R, & \text{for } q = 0, 3, \\ 0, & \text{for } q \neq 0, 3, \end{cases}$$

and to a generator of the group H_{alg}^3 there corresponds a certain exact sequence of representations of the group, containing infinite-dimensional representations.

Received
23 VIII 1967

CITED LITERATURE

1. J. Dixmier, A. Donady, *Bull. Soc. Math. France*, **91**, 227 (1963).
2. J. Dixmier, *J. Math. pure et appl.*, **128**, 1 (1963).
3. I. M. Gel' fand, D. B. Fuks, *DAN*, **176**, No. 1 (1967).
4. S. Mac Lane, *Homology*, IL, 1965.
5. I. M. Gel' fand, M. I. Graev, N. Ya. Vilenkin, *Generalized Functions*, vol. 5, 1962.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.