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Abstract

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MATHEMATICAL PHYSICS

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ON THE PROPERTIES OF THE DECOMPOSITION INTO BUNDLES OF THE CAUSAL S -MATRIX

(Presented by Academician N. N. Bogolyubov on 24 V 1966)

In the present paper we shall prove space-like properties of the decomposition into bundles of vacuum mean values of radiation operators, as well as of matrix elements of the S -matrix within the framework of N. N. Bogolyubov's axiomatic causal S -matrix theory ⁽¹⁾. The properties of the decomposition into bundles of the S -matrix have recently been formulated ⁽²⁾ and proved ⁽³⁾ within the framework of Wightman's axiomatic quantum field theory ⁽⁴⁾. For simplicity we shall consider the self-action of one real scalar field $\varphi(x)$.

Consider

$$S = \sum_{N=0}^{\infty} \frac{(-i)^N}{N!} \int (d^4x)_N h_N(x)_N : \varphi(x_1) \dots \varphi(x_N) :, \quad (1)$$

where the coefficient functions $h_N(x)_N$ are C -numbers with the property $h_N(x_{\alpha N}) = h_N(x_N)$. In formula (1) the following notation has been introduced:

$$(z)_N = (z_1, \dots, z_N), \quad (z_{\alpha})_N = (z_{\alpha_1}, \dots, z_{\alpha_N}),$$

$$\int (d^4x)_N = \int \dots \int d^4x_1 \dots d^4x_N.$$

Following N. N. Bogolyubov ⁽¹⁾, we define the causality condition according to

$$\iint d^4x d^4y f(x)g(y) \frac{\delta}{\delta\varphi(x)} \left(\frac{\delta S}{\delta\varphi(y)} S^+ \right) = 0 \quad (2)$$

for arbitrary $f(x), g(y) \in S(R^4)$, satisfying the condition $f(x)g(y) = 0$ for time-like intervals $(x - y)^2 \geq 0$ and $(x^0 - y^0) \geq 0$.

Let us introduce into consideration the radiation-operator-valued generalized function

$$H_N(f) = \int (d^4x)_N f(x)_N \frac{\delta^N S}{\delta\varphi(x_1) \dots \delta\varphi(x_N)} S^+ =$$

$$= (-i)^N \int (d^4x)_N \theta(x_1^0 - x_2^0) \dots \theta(x_{N-1}^0 - x_N^0) J(x_1) \dots J(x_N) \sum_{p(1, \dots, N)} f(x_\alpha)_N, \quad (3)$$

defined in $D_S \subset H$ for any $f(x)_N \in S_0(R^{4N}) \subset S(R^{4N})$, vanishing together with its derivatives of sufficiently high order when $x_1 = x_2 = \dots = x_N$. The summation in (2) is taken over all possible permutations $\alpha_1, \dots, \alpha_N$ of the numbers $1, 2, \dots$. By $J(x) = i \delta S / \delta\varphi(x)$ we denote the operator of the bosonic current. We shall assume that

$$h_N(f) = i^N \langle 0 | H_N(f) | 0 \rangle \in S'_0(R^{4N}) \quad (4)$$

for every $f(x_1, \dots, x_N) \in S_0(R^{4N})$.

Theorem 1. Let a be an arbitrary space-like vector, $\lambda < 0$. If condition (2) is satisfied, the relation

$$\lim_{\lambda \rightarrow \infty} h_{m+n+r+s}(F_{\lambda a}^{(m+n+r+s)}) = h_{m+n}(f^{(m+n)}) h_{r+s}(g^{(r+s)}) \quad (5)$$

holds for any

$$F_{\lambda a}^{(m+n+r+s)} = F^{(m+n+r+s)}(x, y - \lambda a)_{m+n, r+s} =$$

$$= f^{(m+n)}(x)_{m+n} g^{(r+s)}(y - \lambda a)_{r+s} \in S_0(R^{4(m+n+r+s)}), \quad (6)$$

where $f^{(m+n)}(x)_{m+n}, g^{(r+s)}(y)_{r+s} \in S_0(R^{4(m+n)}, R^{4(r+s)})$.

One can indicate a relation analogous to (5) for matrix elements of the S -matrix.

Between the matrix elements $S_{m+r, n+s}(\mathbf{q}', \mathbf{p}'; \mathbf{q}, \mathbf{p})_{m, r; n, s} \in S'(R^{3(m+n+r+s)})$ and the vacuum averages of radiation operators $h_{m+n+r+s}(x', y'; x, y)_{m, r; n, s} \in S'(R^{4(m+n+r+s)})$ there holds the relation

$$S_{m+r, n+s} \left(\exp \left[-i\lambda \left(\sum_{j=1}^r a \cdot p_j - \sum_{k=1}^s a \cdot p_{k+r} \right) \right] \tilde{f}^{(m+n)} \tilde{g}^{(r+s)} \right) = h_{m+n+r+s}(f^{(m+n)} g_{\lambda a}^{(r+s)}), \quad (7)$$

where

$$f^{(m+n)}(x)_{m+n} = \frac{1}{2\pi^{3(m+n)/2}} \int \frac{\exp \left[i \left(\sum_{l=1}^m q_l x_l - \sum_{t=1}^n q_{m+t} x_{m+t} \right) \right]}{\sqrt{2q_1^0 \dots 2q_{m+n}^0}} \tilde{f}^{(m+n)}(\mathbf{q})_{m+n} (dq)_{m+n},$$

$$q_i^0 = \sqrt{\mathbf{q}_i^2 + m^2}, \quad i = 1, \dots, m+n, \quad (8)$$

where $\tilde{f}^{(m+n)}(\mathbf{q})_{m+n} \in S(\tilde{R}^{3(m+n)})$ and $f^{(m+n)}(R^{4(m+n)})$,

The function $g^{(r+s)}(y)_{r+s} \in S(R^{3(r+s)})$ has the same structure as $f^{(m+n)}(x)_{m+n}$, and $g_{\lambda a}^{(r+s)} = g^{(r+s)}(y - \lambda a)_{r+s}$, where a is an arbitrary space-like vector, $\lambda > 0$.

Theorem 2. Suppose the condition of Theorem 1 is fulfilled. Then

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} S_{m+r, n+s} \left(\exp \left[-i\lambda \left(\sum_{j=1}^r a \cdot p_j - \sum_{k=1}^s a \cdot p_{r+k} \right) \right] \tilde{f}^{(m+n)} \tilde{g}^{(r+s)} \right) = \\ = S_{m+n}(\tilde{f}^{(m+n)}) S_{r+s}(\tilde{g}^{(r+s)}) \end{aligned} \quad (9)$$

for any $\tilde{f}^{(m+n)}(\mathbf{q}_{m+n}) \in S_0(\tilde{R}^{3(m+n)})$ and $\tilde{g}^{(r+s)}(\mathbf{p}_{r+s}) \in S_0(\tilde{R}^{3(r+s)})$.

It is obvious that the proof of Theorem 2 follows directly from Theorem 1. We divide the proof of the theorem into the proofs of several lemmas.

Lemma 1. Suppose the causality condition (1) is fulfilled. Then

$$H_{m+n+r+s}(F^{(m+n+r+s)}) = H_{m+n}(f^{(m+n)}) H_{r+s}(g^{(r+s)}) \quad (10)$$

for any function

$$F^{(m+n+r+s)}(x, y)_{m+n, r+s} \in S_0(R^{4(m+n+r+s)}),$$

representable in the form

$$F^{(m+n+r+s)}(x, y)_{m+n, r+s} = f^{(m+n)}(x)_{m+n} g^{(r+s)}(y)_{r+s}, \quad (11)$$

provided $\{x\}_{n+m} \geq \{y\}_{r+s}$, where $f^{(m+n)}(x)_{m+n} \in S_0(R^{4(m+n)})$ and $g^{(r+s)}(y)_{r+s} \in S_0(R^{4(r+s)})$.

The proof of the lemma is based on the results of work (5).

Now, using this lemma and the completeness condition for the system of eigen-amplitudes of the 4-energy-momentum operator $\hat{P}^\mu$, we obtain

$$h_{m+n+r+s}(F_a^{(m+n+r+s)}) = h_{m+n}(f^{(m+n)}) h_{r+s}(g^{(r+s)}) + h_{m+n+r+s}^T(F_{\lambda a}^{(m+n+r+s)}), \quad (12)$$

where $h_{m+n+r+s}^T(F_{\lambda a}^{(m+n+r+s)})$, the truncated vacuum mean value of the radiation operators, is defined according to

$$\begin{aligned} h_{m+n+r+s}^T(F_{\lambda a}^{(m+n+r+s)}) = & \iint (d^4x)_{m+n-1} (d^4y)_{r+s} \theta(x_1^0 - x_2^0) \cdots \\ & \cdots \theta(x_{m+n-1}^0 - x_{m+n}^0) \theta(y_1^0 - y_2^0) \cdots \theta(y_{r+s-1}^0 - y_{r+s}^0) \\ & \times \left[\sum_{\nu=1}^{\infty} \frac{1}{\nu!} \int (dk)_\nu \langle 0 | J(x_1) \cdots J(x_{m+n}) | (k)_\nu \rangle \langle (k)_\nu | J(y_1) \cdots J(y_{r+s}) | 0 \rangle \right] \\ & \times \sum_{P(1, \dots, m+n)} f^{(m+n)}(x_\alpha)_{m+n} \sum_{P(1, \dots, r+s)} g^{(r+s)}(y_\beta - \lambda a)_{r+s} \end{aligned} \quad (13)$$

for arbitrary $f^{(m+n)}(x)_{m+n} \in S_0(R^{4(m+n)})$ and $g^{(r+s)}(y)_{r+s} \in S_0(R^{4(r+s)})$.

Lemma 2. The Fourier transform of $h_{m+n+r+s}^T(F_{\lambda a}^{(m+n+r+s)})$ belongs to the space $S_0(\tilde{R}^{3(m+n+r+s)})$.

Indeed, using the completeness properties of the amplitude of the 4-energy-momentum operator $\hat{P}^\mu$, from (11) we obtain

$$\begin{aligned} h_{m+n+r+s}^T(F_{\lambda a}^{(m+n+r+s)}) = & i^{m+n+r+s} \int \cdots \int I(\dots, \mu_{\alpha 3}^2, \dots; \dots, m_{ij}^2, \dots; M^2) \\ & \times \tilde{D}_{0, \mu_{\alpha 3}^2, m_{ij}^2, M^2}^{(-)}(e^{i\lambda a k} \tilde{f}_2^{(m+n)} \tilde{g}_2^{(r+s)}) \\ & \times \prod_{\alpha \leq \beta \leq 1}^{m+n+1} d\mu_{\alpha \beta}^2 \prod_{i < j \leq 1}^{r+s+1} dm_{ij}^2 dM^2, \end{aligned} \quad (14)$$

where

$$\tilde{D}_{0, \mu_{\alpha 3}^2, m_{ij}^2, M^2}^{(-)} \in S'(R(\mu_{\alpha 3}^2, m_{ij}^2, M^2))$$

and is a Lorentz-invariant generalized function. Since the carrier of the generalized function $I(\dots, \mu_{\alpha \beta}^2, \dots; \dots, m_{ij}^2, \dots; M^2) \in S'(R)$ lies in the region

$$\{\dots, \mu_{\alpha \beta}^2, \dots; \dots, m_{ij}^2, \dots; M^2\} > 0, \quad (15)$$

the integral (14) exists, and therefore $h_{m+n+r+s}^T(F_{\lambda a}^{(m+n+r+s)})$ depends linearly and continuously on the functions $\tilde{f}_2(q, k)_{m+n-1,0} \in S_0(\tilde{R}^{3(m+n)})$ and $\tilde{g}_2^{(r+s)}(p, k)_{r+s-1,0} \in S_0(\tilde{R}^{3(r+s)})$.

Lemma 3. Let a be an arbitrary spacelike vector, $\lambda > 0$. Then

$$\lim_{\lambda \rightarrow \infty} \lambda^N h_{m+n+r+s}^T(F_{\lambda a}^{(m+n+r+s)}) = 0 \quad (16)$$

for any $F^{(m+n+r+s)} \in S_0(R^{4(m+n+r+s)})$ of the form (5).

We note that the proof of this lemma reduces, on the basis of formula (13), to proving the relation

$$\lim_{\lambda \rightarrow \infty} \lambda^N \tilde{D}_{0, \mu_{\alpha\beta}^2, m_{ij}^2, M^2}^{(-)}(e^{i\lambda a k'} \tilde{f}_2^{(m+n)} \tilde{g}_2^{(r+s)}) = 0 \quad (17)$$

for $\tilde{f}_2^{(m+n)}(q, k)_{m+n-1,0} \in S_0(\tilde{R}^{3(m+n)})$ and $\tilde{g}_2^{(r+s)}(p, k)_{r+s-1,0} \in S_0(\tilde{R}^{3(r+s)})$.

In (16) the reference frame $d \equiv (0, a, 0, 0)$ is chosen, and k' is the component of the vector $k = (k^0, k^1, k^2, k^3)$. Araki [6] proved that relation (17) holds in the case when $\tilde{D}_{0, \mu_{\alpha\beta}^2, m_{ij}^2, M^2}^{(-)}$ is a Lorentz-invariant generalized function. Since in the present case $\tilde{D}_{0, \mu_{\alpha\beta}^2, m_{ij}^2, M^2}^{(-)}$ is a Lorentz-invariant generalized function, Lemma 3 is thereby proved.

On the basis of the lemmas, Theorem 1 is completely proved and, consequently, Theorem 2 as well.

Now define the extension of the radiation operator $H_N(f)$, given by formula (3) in $S_0(R^{4N})$, to the whole space $S(R^{4N})$ according to

$$\begin{aligned} H_N^C(f) &= \int (d^4x)_N \theta(x_1^0 - x_2^0) \cdots \theta(x_{N-1}^0 - x_N^0) J(x_1) \cdots J(x_N) \times \\ &\times \sum_{P(1, \dots, N)} f(x_\alpha)_N + \int (d^4x)_N f(x)_N C_N^\Lambda(x)_N \end{aligned} \quad (18)$$

for any function $f(x)_N \in S(R^{4N})$, where $C_N^\Lambda(x)_N$ is expressed as a sum of symmetrized T -products of chains of quasilocal operators $\Lambda_n(x_1, \dots, x_n)$ (5), possessing the properties of locality, Hermiticity, symmetry, and local commutativity.

Since

$$h_N^C(f) = i^N \langle 0 | H_N^C(f) | 0 \rangle \in S'(R^{4N}) \quad (19)$$

for any $f \in S(R^{4N})$, the properties of the cluster decomposition of the vacuum mean values of the radiation operators with coinciding arguments will be defined according to

$$\lim_{\lambda \rightarrow \infty} h_{m+n+r+s}^C(F_{\lambda a}) = h_{m+n}^C(f)h_{r+s}^C(g) \quad (20)$$

for any function $F(x, y)_{m+n, r+s} = f(x)_{m+n}g(y)_{r+s} \in S(R^{4(m+n+r+s)})$, where $f(x)_{m+n} \in S(R^{4(m+n)})$ and $g(y)_{r+s} \in S(R^{4(r+s)})$, with causally independent carriers $\{x\}_{m+n} > \{y\}_{r+s}$.

Extending relation (7) to the whole space $S(R^{4(m+n+r+s)})$ and using formula (20), we obtain the cluster-decomposition properties of the elements of the S -matrix, expressed in the form of formula (9).

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