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Abstract

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MATHEMATICS

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ON A NONELLIPTIC DIRICHLET PROBLEM

(Presented by Academician S. L. Sobolev on 20 II 1967)

Consider an elliptic system of two differential equations with real coefficients, written in complex form as a single equation

$$w_{\bar{z}\bar{z}} - A_1 \bar{w}_z - A_2 w_z - A_3 \bar{w}_{\bar{z}} - A_4 w_{\bar{z}} - A_5 \bar{w} - A_6 w = F, \quad (1)$$

where $w = u_1 + iu_2$; $\bar{w} = u_1 - iu_2$; $\partial/\partial z = \frac{1}{2}(\partial/\partial x - i\partial/\partial y)$, $\partial/\partial \bar{z} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$; $F(z, \bar{z})$, $A_j(z, \bar{z})$ ($j = 1, \dots, 6$) are analytic functions of the variables z and $\bar{z}$ in some cylindrical domain (T, T^*) , T being some domain of the plane $z = x + iy$, and T^* the reflection of the domain T with respect to the real axis.

Every twice continuously differentiable solution of equation (1) in the domain T is an analytic function $(1,2)$.

For equation (1) the first boundary-value problem is studied in the following formulation:

Find a solution of equation (1) in a bounded simply connected domain $D \subset T$ with boundary $L \in A^{(2,\alpha)}$, belonging to the class $C^2(D) \cap C^{(1,\alpha)}(D + L)$, under the condition

$$w|_L = f, \quad (2)$$

where f is a given complex function of class $C^{(k,\alpha)}(a)$, and $k \geq 2$.

In the absence of lower-order terms ($A_j = 0$, $j = 1, \dots, 6$), problem (2)–(1) is not Noetherian.* For example, in the disk $|z - z_0| \leq R$ the homogeneous problem has an infinite number of linearly independent solutions (3) . However, problem (1)–(2) becomes Noetherian when terms containing $\bar{w}$ are added.

Theorem 1. *The Dirichlet problem for the equation*

$$\partial^2 w / \partial \bar{z}^2 - A_1(z, \bar{z}) \partial \bar{w} / \partial z = 0 \quad (3)$$

is Noetherian if

$$A_1(z, \bar{z}) \neq 0 \quad \text{on } L,$$

and the index of the problem is equal to

$$\frac{1}{\pi} [\arg A_1(z, \bar{z})]_L - 2,$$

where $[]_L$ denotes the increment of the argument of the function $A_1(z, \bar{z})$ when the domain D is traversed once in the positive direction.

Proof. Equation (3) is equivalent to the system

$$\partial w / \partial \bar{z} - u = 0,$$

$$\partial u / \partial \bar{z} - A_1 \bar{u} = 0, \quad (4)$$

* Here problem (1)–(2) is called Noetherian if the homogeneous problem has a finite number of linearly independent solutions, and the nonhomogeneous problem is solvable subject to a finite number of orthogonality conditions on the function f .

which is equivalent to the system of Volterra integral equations

$$w - \int_0^{\bar{z}} u(z, \tau) d\tau = \psi(z), \quad u - \int_0^{\bar{z}} A_1(z, \tau) \overline{u(\bar{z}, \bar{\tau})} d\tau = \varphi''(z), \quad (5)$$

where $\psi(z)$ and $\varphi''(z)$ are arbitrary functions holomorphic in the domain D , with $\varphi(0) = \varphi'(0) = 0$.

Solving the system (5), we obtain a general representation of an arbitrary solution, regular in the domain D , of equation (3) in terms of two arbitrary holomorphic functions $\varphi(z)$ and $\psi(z)$:

$$\begin{aligned} w = & \psi + z\varphi'' + A_0\bar{\varphi} + B\varphi' + C\varphi + \\ & + \int_0^z F_1(z, \bar{z}, t)\varphi(t) dt + \int_0^{\bar{z}} F_2(z, \bar{z}, \tau)\overline{\varphi(\bar{\tau})} d\tau, \end{aligned} \quad (6')$$

where $B, C(z, \bar{z}), F_1(z, \bar{z}, t), F_2(z, \bar{z}, \tau)$ are analytic functions of their arguments and are expressed through the coefficient A_1 ; moreover, from (5) it follows that if $w \in C^{(1,\alpha)}(D + L)$, then $\varphi(z) \in C^{(2,\alpha)}(D + L)$, $\psi(z) \in C^{(0,\alpha)}(D + L)$. Using the integral representations

$$\varphi(z) = \frac{1}{\pi i} \int_L \frac{\mu_1(t) dt}{t-z} + iC_1, \quad \psi(z) = \frac{1}{\pi i} \int_L \frac{\mu_2(t) dt}{t-z} + iC_2, \quad (7)$$

where μ_1, μ_2 are real densities $\mu_1(t) \in C^{(2,\alpha)}(L)$, $\mu_2 \in C^{(0,\alpha)}(L)$, and C_1, C_2 are real constants ⁽⁴⁾, the solution of equation (3) can be written in the form

$$\begin{aligned} w(z, \bar{z}) = & \frac{1}{\pi i} \int_L \frac{\mu_2(t) + B(t, \bar{t})\mu_1'(t) + \bar{t}\mu_1''}{t-z} dt - \frac{1}{\pi i} \int_L \frac{\bar{t}-\bar{z}}{t-z} \mu_1'(t) dt \\ & - \frac{1}{\pi i} \int_L \frac{B(z, \bar{t}) - B(z, \bar{z})}{t-z} \mu_1'(t) dt - \frac{A_1(z, \bar{z})}{\pi i} \int_L \frac{\mu_1(t) d\bar{t}}{\bar{t}-\bar{z}} + \frac{C}{\pi i} \int_L \frac{\mu_1 dt}{t-z} \\ & + \frac{1}{\pi i} \int_L \mu_1(t) \int_0^z \frac{F_1(z, \bar{z}, \xi) d\xi}{t-\xi} dt + \frac{1}{\pi i} \int_L \mu_1(\bar{t}) \int_0^{\bar{z}} \frac{F_2(z, \bar{z}, \eta) d\eta}{\bar{t}-\eta} d\bar{t} + iC_2. \end{aligned} \quad (8)$$

The first integral in formula (8) is a holomorphic function of class $C^{(0,\alpha)}(D+L)$ and can be represented by a Cauchy-type integral with real density $\nu(t) \in C^{(0,\alpha)}(L)$ in the form

$$\frac{1}{\pi i} \int_L \frac{\mu_2(t) + B(z, \bar{t})\mu_1' + \bar{t}\mu_1''}{t-z} dt + iC_2 = \frac{1}{\pi i} \int_L \frac{\nu(t) dt}{t-z} + iC_3, \quad (9)$$

where $\nu(t)$ and C_3 are uniquely determined through μ_1, μ_2, C_2 . Conversely, knowing μ_1, ν, C_3 , one can uniquely determine μ_2, C_2 .

Substituting (9) into formula (8), we obtain a representation of an arbitrary solution w of problem (2)–(3) in terms of two arbitrary real functions $\mu_1(t)$ and $\nu(t)$:

$$\begin{aligned} w(z, \bar{z}) = & \frac{1}{\pi i} \int_L \frac{\nu(t) dt}{t-z} + iC_3 + \frac{C(z, \bar{z})}{\pi i} \int_L \frac{\mu_1(t) dt}{t-z} - \frac{A_1(z, \bar{z})}{\pi i} \int_L \frac{\mu_1(t) d\bar{t}}{\bar{t}-\bar{z}} \\ & - \frac{1}{\pi i} \int_L \frac{\bar{t}-\bar{z}}{t-z} \mu_1'(t) dt - \frac{1}{\pi i} \int_L \frac{B(z, \bar{t}) - B(z, \bar{z})}{t-z} \mu_1'(t) dt \\ & + \frac{1}{\pi i} \int_L \mu_1(t) \int_0^z \frac{F_1(z, \bar{z}, \xi) d\xi}{t-\xi} dt + \frac{1}{\pi i} \int_L \mu_1(\bar{t}) \int_0^{\bar{z}} \frac{F_2(z, \bar{z}, \eta) d\eta}{\bar{t}-\eta} d\bar{t}. \end{aligned} \quad (10)$$

Passing to the limit as $z \rightarrow t_0 \in L$, we obtain a singular integral equation of the form

$$\nu(t_0) + \frac{1}{\pi i} \int_L \frac{\nu(t) dt}{t-t_0} + A_1(t_0, \bar{t}_0)\mu_1(t_0) - \frac{A_1(t_0, \bar{t}_0)}{\pi i} \int_L \frac{\mu_1 d\bar{t}}{\bar{t}-\bar{t}_0} +$$

$$\begin{aligned}
& + C(t_0, \bar{t}_0) \mu_1(t_0) + \frac{C(t_0, \bar{t}_0)}{\pi i} \int_L \frac{\mu_1 dt}{t - t_0} + \int_L K_1(t_0, \bar{t}_0, t) \mu_1(t) dt + \\
& + \int_L K_2(t_0, \bar{t}_0, \bar{t}) \mu_1(\bar{t}) d\bar{t} = f(t_0) - iC_3,
\end{aligned} \tag{11}$$

where the kernels K_1 and K_2 have at $t = t_0$ a singularity of order less than 1. Equating the real and imaginary parts of expression (11), we obtain a system of two equations with real coefficients. It is easy to see that, for $A_1(t_0, \bar{t}_0) \neq 0$, this system is of normal type (4).

Equation (11) is equivalent to problem (3)–(2), i.e., if problem (3)–(2) has a solution, then equation (11) is solvable, and, conversely, if equation (11) is solvable, then problem (3)–(2) has a solution.

Since $\varphi(0) = \varphi'(0) = 0$, we obtain the conditions on $\mu_1(t)$

$$\frac{1}{\pi i} \int_L \frac{\mu_1(t) dt}{t} + iC_1 = 0, \quad \frac{1}{\pi i} \int_L \frac{\mu_1(t)}{t^2} dt = 0. \tag{12}$$

Problem (3)–(2) is equivalent to equation (11) and to condition (12), which is used only in computing the index. We note that for $f \in C^{(2,\alpha)}(L)$ all Hölder-continuous solutions (μ_1, ν) of equation (11), for $A_1 \neq 0$ on L , belong to the class $C^{(2,\alpha)}(L)$, and, according to formula (10), the solution of problem (2)–(3) belongs to the class $C^{(1,\alpha)}(D + L)$, but it may fail to belong not only to $C^{(2,\alpha)}(D + L)$, but even to $C^{(1,\beta)}(D + L)$ for $\beta > \alpha$. Indeed, in formula (10) all integrals, except the second, belong to $C^{(2,\alpha)}(D + L)$, while the second integral

$$\eta(x, y) = -\frac{1}{\pi i} \int_L \frac{\bar{t} - \bar{z}}{t - z} \mu_1''(t) dt$$

is a function of class $C^{(1,\alpha)}(D + L)$, when $\mu_1(t) \in C^{(2,\alpha)}(L)$ (4).

For the disk $|z| \leq 1$, when $A_1 = \text{const} \neq 0$, a more concrete result is obtained. The general representation (6) in this case has the form:

$$\begin{aligned}
w(z, \bar{z}) = & \psi(z) + \bar{z}\varphi(z) + A_1 \int_0^{\bar{z}} \overline{(z-t)\varphi(t)} dt + \\
& + |A_1|^2 \int_0^z \varphi(t) \left\{ \int_0^z d\tau_1 \int_0^{\tau_1} I_0(2|A_1|(z-t)^{1/2}(\tau_1-\tau)^{1/2}) d\tau \right\} dt +
\end{aligned}$$

$$+A_1|A_1|^2 \int_0^{\bar{z}} d\xi \int_0^\xi d\tau \left[\int_0^z I_0(2|A_1|(z-t)^{1/2}(\xi-\tau)^{1/2}) dt \cdot \int_0^{\bar{\tau}} \overline{\varphi(\eta)} d\eta \right], \quad (6'')$$

where I_0 is the Bessel function of zero order of imaginary argument.

Expanding the functions $\varphi(z)$ and $\psi(z)$ in Taylor series,

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \psi(z) = \sum_{n=0}^{\infty} b_n z^n,$$

and $f(\theta)$ in a Fourier series,

$$f(\theta) = \sum_{n=-\infty}^{+\infty} f_n e^{in\theta}, \quad \text{where} \quad f_n = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{in\theta} d\theta,$$

and sub-

stituting into formula (6), for $|z| = 1$ we obtain a system of linear equations for determining the coefficients $a_n, b_n, n \geq 0$

$$I_1(2|A_1| - 1)a_0 = f_1, \quad A_1^{-1}I_2(2|A_1|)a_0 = \bar{f}_{-2},$$

$$|A_1|^{-n-2} \bar{A}_1 n! I_{n+2}(2|A_1|)a_n = \bar{f}_{-(n+2)} \quad (n \geq 1),$$

$$b_n + a_{n+1} [|A_1|^{-n-2} (n+1)! I_n(2|A_1|) - (n+1)! |A_1|^{-2} - 1] = f_n \quad (n \geq 0),$$

where

$$I_n(2|A_1|) = \sum_{k=0}^{\infty} \frac{|A_1|^{2k+n}}{k!(k+n)!},$$

whence we obtain:

Theorem 2. The homogeneous problem (3')–(2) for $A_1 = \text{const} \neq 0$ in the disk $|z| \leq 1$ has only the trivial solution, and the nonhomogeneous problem is solvable under the condition

$$I_2(2|A_1|) \int_0^{2\pi} \bar{f}(\theta) e^{i\theta} d\theta = \bar{A}_1 (I_1(2|A_1|) - 1) \int_0^{2\pi} f(\theta) e^{-2i\theta} d\theta.$$

By analogous methods one obtains:

Theorem 3. The Dirichlet problem for the equation

$$w_{z\bar{z}} - (A_3 \bar{w})_{\bar{z}} = 0 \quad (13)$$

is Noetherian if

$$A_3(z, \bar{z}) \neq 0 \quad \text{on } L,$$

and the index κ of the problem is equal to

$$\frac{1}{\pi} [\arg A_3(z, \bar{z})]_L + 2.$$

Theorem 4. The homogeneous problem (13)–(2) for $A_3 = \text{const} \neq 0$ in the disk $|z| \leq 1$ has two nontrivial linearly independent solutions over the field of real numbers, and the nonhomogeneous problem is unconditionally solvable for $f \in C^{(2,\alpha)}(L)$.

Theorem 5. The Dirichlet problem for the equation

$$w_{z\bar{z}} - A_5 \bar{w} = 0 \quad (14)$$

is Noetherian if $A_5(z, \bar{z}) \neq 0$ on L and $f \in C^{(3,\alpha)}(L)$, and the index of the problem is equal to

$$\frac{1}{\pi} [A_5(z, \bar{z})]_L.$$

It is easy to see that the Dirichlet problem for the equation $w_{z\bar{z}} - A_4 w_z = 0$, $A_4 = \text{const} \neq 0$, in the disk $|z| \leq 1$, is not Noetherian, since the general representation in this case has the form $w = \psi(z) + \varphi(z)e^{A_4 z}$, where φ, ψ are arbitrary holomorphic functions in the disk.

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References

1. I. G. Petrovskii, *Matem. sborn.*, **5** (47), 1, 3 (1939).
2. I. N. Vekua, *Matem. sborn.*, **31** (73), 2, 219 (1952).
3. A. V. Bitsadze, *Boundary value problems for elliptic equations of second order*, "Nauka," 1966.
4. N. I. Muskhelishvili, *Singular integral equations*, Moscow, 1962.
5. I. N. Vekua, *New methods for solving elliptic equations*, Moscow, 1948.

Note: Figure translations are in progress. See original paper for figures.

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