

ON BOUNDARY PROPERTIES OF MEROMORPHIC FUNCTIONS OF THE CLASS $\Sigma(N, \alpha)$

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Abstract

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MATHEMATICS

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ON BOUNDARY PROPERTIES OF MEROMORPHIC FUNCTIONS OF THE CLASS N_α

1°. In the papers ^(1,2), for each function $F(z)$ meromorphic in the disk $|z| < 1$, and for any value of the parameter α ($-1 < \alpha < +\infty$), three functions $m_\alpha(r; F)$, $N_\alpha(r; F)$, and $T_\alpha(r; F) \equiv m_\alpha(r; F) + N_\alpha(r; F)$ were defined, which are peculiar analogues of the well-known Nevanlinna functions $m(r; F)$, $N(r; F)$, and $T(r; F)$, and coincide with them when $\alpha = 0$. The class N_α was defined by means of the function $T_\alpha(r; F)$ (also called the α -characteristic) as the set of those $F(z)$ for which

$$T^{(\alpha)}(F) = \sup_{0 < r < 1} \{T_\alpha(r; F)\} < +\infty. \quad (1)$$

Since $T_0(r; F) = T(r; F)$, it follows that $N_0 = N$, where N is the class of functions of bounded type of R. Nevanlinna ⁽³⁾.

As is known ⁽³⁾, if $F(z) \in N = N_0$, then the limit

$$F(e^{i\vartheta}) = \lim_{r \rightarrow 1-0} F(re^{i\vartheta}) \quad (2)$$

exists everywhere on $[0, 2\pi]$, except, perhaps, for a certain exceptional set $E \subset [0, 2\pi]$ of linear measure zero.

In view of the fact that the classes N_α ($-1 < \alpha < +\infty$) decrease monotonically as α decreases ^(1,2), and, in particular, $N^\alpha \subset N_0$ ($-1 < \alpha < 0$), the problem naturally arises of the possibility of a finer characterization of the exceptional set $E \subset [0, 2\pi]$ where the limit (2) for $F(z) \in N_\alpha$ ($-1 < \alpha < 0$) may fail to exist. In the present note a solution of this problem is given by using the notion of γ -capacity of sets (see, for example, ⁽⁴⁾).

The following basic theorem is proved.

Theorem 1. If $F(z) \in N_\alpha$ ($-1 < \alpha < 0$), then the limit (2) exists everywhere on $[0, 2\pi]$, except, perhaps, for a certain set E , whose γ -capacity (where $1 + \alpha < \gamma < 1$ is any number) is equal to zero.

Below we shall give a proof of this theorem, as well as three auxiliary theorems which are also of independent interest.

An important example of a function of the class N_α with zeros at the points of a given sequence $\{z_k\}_1^\infty$ ($0 < |z_k| \leq |z_{k+1}| < 1$), subject only to the condition

$$\sum_{k=1}^{\infty} (1 - |z_k|)^{1+\alpha} < +\infty, \quad (3)$$

is the product, convergent in the disk $|z| < 1$, ^(1,2)

$$B_\alpha(z; z_k) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) e^{-W_\alpha(z; z_k)}, \quad (4)$$

where, for $|z| < 1$ and $|\zeta| < 1$,

$$W_\alpha(z; \zeta) = \int_{|\zeta|}^1 \frac{(1-x)^\alpha}{x} dx - \sum_{k=1}^{\infty} \frac{\Gamma(1+\alpha+k)}{\Gamma(1+\alpha)\Gamma(1+k)} \left\{ \xi^{-k} \int_0^{|\xi|} (1-x)^\alpha x^{k-1} dx - \bar{\xi}^k \int_{|\xi|}^1 (1-x)^\alpha x^{-k-1} dx \right\} z^k. \quad (5)$$

The proof of Theorem 1 rests essentially on the parametric representation of the class N_α ^(1, 2) and on the well-known theorem of Salem and Zygmund ⁽⁴⁾.

Theorem A. *The class N_α ($-1 < \alpha < \infty$) coincides with the set of functions admitting the representation*

$$F(z) = cz^\lambda \frac{B_\alpha(z; a_\mu)}{B_\alpha(z; b_\nu)} \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} S_\alpha(e^{-i\vartheta} z) d\psi(\vartheta) \right\}, \quad (6)$$

where c is a constant, $\lambda \geq 0$ is an integer,

$$S_\alpha(z) = \Gamma(1+\alpha) \left\{ \frac{2}{(1-z)^{1+\alpha}} - 1 \right\}, \quad \operatorname{Re} S_\alpha(z) \geq 0, \quad (7)$$

$\psi(\vartheta)$ is an arbitrary real function of finite total variation on $[0, 2\pi]$.

Theorem B ⁽⁴⁾. *If*

$$\sum_{k=1}^{\infty} (a_k^2 + b_k^2) k^\beta < +\infty \quad (0 < \beta < 1),$$

then the trigonometric series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

can diverge only on a set whose $(1 - \beta)$ -capacity is zero.

We note that before the appearance of $(^1, ^2)$, other classes T_β ($0 < \beta < 1$) included in the class N were known $(^5)$: $w(z) \in T_\beta$ if

$$\int_0^1 \frac{A(r; w)}{(1-r)^\beta} dr < +\infty, \quad \text{where} \quad A(r; w) = \iint_{|z| < r} \frac{|w'(z)|^2}{(1 + |w(z)|^2)^2} dx dy. \quad (8)$$

From this definition it follows directly that

Lemma C. *If the function $w(z) = \sum_{k=0}^{\infty} a_k z^k$ ($|z| < 1$) is bounded and belongs to the class T_β ($0 < \beta < 1$), then*

$$\sum_{k=1}^{\infty} |a_k|^2 k < +\infty.$$

In his study devoted to the classes T_β , L. Carleson first proved that for $F(z) \in T_\beta$ ($0 < \beta < 1$) the assertion of Theorem 1 is valid, with the value $\gamma = 1 - \beta$.*

2°. Let us observe that for $\alpha = 0$ the function $B_\alpha(z; z_k)$ coincides with the ordinary Blaschke product, i.e.

$$B_0(z; z_k) = B(z) = \prod_{k=1}^{\infty} \frac{z_k - z}{1 - \bar{z}_k z} \left| \frac{z_k}{z} \right|. \quad (9)$$

We first give a theorem that plays an important role in the theorems set forth below.

Theorem 2. *If $-1 < \alpha < 0$ and condition (4) is satisfied, then*

$$|B_\alpha(z; z_k)| \leq |B_0(z; z_k)| < 1 \quad (|z| < 1). \quad (10)$$

* If the function $F(z) \in T_\beta$ is bounded, then this theorem follows directly from Theorem B and Lemma C. In the general case, since there is no representation for the classes T_β , L. Carleson established this theorem by a more complicated method. This explains also the fact that not every function $F \in T_\beta$ can be represented in the form of a quotient f/g of bounded functions from the same class $(^5)$ —a “defect” of which the class N_α is free.

Let us first note that, by virtue of the definition (6) of the function $B_\alpha(z; z_k)$, it is enough to establish the inequality

$$U_\alpha(z; \zeta) \equiv \operatorname{Re}\{W_\alpha(z; \zeta) - W_0(z; \zeta)\} \geq 0 \quad (|z| < 1, |\zeta| < 1). \quad (11)$$

But from the expansion (5) of the function $W_\alpha(z; \zeta)$ we have

$$U_\alpha(z; \zeta) = \frac{a_0(\rho)}{2} + \sum_{k=1}^{\infty} a_k(\rho) |\omega|^k \cos(k \arg \omega), \quad (12)$$

where $\rho = |\zeta|$, $\omega = \bar{\zeta}z$ ($|\omega| = \rho|z| < 1$),

$$a_0(\rho) = 2 \int_{\rho}^1 [(1-x)^\alpha - 1] x^{-1} dx, \quad (13)$$

$$a_k(\rho) = \frac{1}{k} - \frac{\Gamma(1+\alpha+k)}{\Gamma(1+\alpha)\Gamma(1+k)} \left\{ \rho^{-2k} \int_0^{\rho} (1-x)^\alpha x^{k-1} dx - \int_{\rho}^1 (1-x)^\alpha x^{-k-1} dx \right\}$$

$$(k = 1, 2, \dots).$$

Let us note further that it is enough to establish the convexity of the sequence $\{a_k(\rho)\}_1^{\infty}$, i.e., the validity of the inequalities

$$\varphi_k(\rho) \equiv \Delta^2 a_k(\rho) \equiv a_k(\rho) - 2a_{k+1}(\rho) + a_{k+2}(\rho) \geq 0$$

$$(0 < \rho \leq 1; k = 0, 1, 2, \dots), \quad (14)$$

so that (11) would follow from this by the known theorem ⁽⁴⁾ on nonnegative trigonometric series. The proof of the inequalities (14) is by no means simple, but we shall only outline it for lack of space. First, we consider the function

$$\rho^5 \varphi_0'(\rho) = 2(1-\rho^2)^2 [1 - (1-\rho)^\alpha] + \alpha \rho (1-\rho)^\alpha [4\rho^2 - (1+\alpha)\rho - 2]$$

($0 < \rho \leq 1$), and by means of appropriate calculations and estimates we verify that all coefficients of its expansion in powers of ρ are negative. Since $\varphi_0(1) = 0$, it follows from this that $\varphi_0(\rho) \geq 0$ ($0 < \rho \leq 1$).

To prove the inequalities (14) for $k \geq 1$, one must first write the functions $a_k(\rho)$ ($k \geq 1$) in the form $a_k(\rho) = b_k(\rho) + d_k(\rho)$, where

$$b_k(\rho) = \frac{1}{k} - \frac{\Gamma(1+\alpha+k)}{\Gamma(1+\alpha)\Gamma(1+k)} \rho^{-2k} \int_0^{\rho^2} (1-x)^\alpha x^{k-1} dx,$$

$$d_k(\rho) = \frac{\Gamma(1+\alpha+k)}{\Gamma(1+\alpha)\Gamma(1+k)} \left\{ \int_{\rho}^1 (1-x)^{\alpha} x^{-k-1} dx - \rho^{-2k} \int_{\rho^2}^{\rho} (1-x)^{\alpha} x^{k-1} dx \right\},$$

noting that $b_k(\rho) \geq b_k(1) = 0$, $d_k(\rho) \geq d_k(1) = 0$ ($k = 1, 2, \dots$). Then, by direct estimates, the convexity of both sequences $\{b_k(\rho)\}_1^{\infty}$ and $\{d_k(\rho)\}_1^{\infty}$ is established. Finally, since

$$\varphi_k(\rho) = \Delta^2 b_k(\rho) + \Delta^2 d_k(\rho), \quad k = 1, 2, \dots,$$

it follows that $\varphi_k(\rho) \geq 0$ ($0 < \rho \leq 1$; $k = 1, 2, \dots$).

It is known ⁽⁵⁾ that, under condition (3), $B(z) \in T_{-\alpha}$. Relying on Theorem 2 and on this fact, one proves

Theorem 3. *If the sequence $\{z_k\}_1^{\infty}$ satisfies condition (3) for $-1 < \alpha < 0$, then $B_{\alpha}(z; z_k) \in T_{-\alpha}$.*

3°. Consider the generalized Cauchy-Stieltjes integral

$$K_{\alpha}(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\psi(\vartheta)}{(1 - e^{-i\vartheta} z)^{1+\alpha}} \quad (-1 < \alpha < 0), \quad (15)$$

where $\psi(\vartheta)$ is an arbitrary real function of bounded variation on $[0, 2\pi]$.

Theorem 4. If $\psi(\vartheta)$ is nondecreasing on $[0, 2\pi]$, then the function

$$w(z) = \exp\{-K_{\alpha}(z)\} \quad (16)$$

is bounded in the disk $|z| < 1$ and, for any α' ($-1 < \alpha < \alpha' < 0$), belongs to the class $T_{-\alpha'}$.

Let us outline the proof of this theorem. Since, for $|z| < 1$,

$$\operatorname{Re} \left[\frac{1}{(1 - ze^{-i\vartheta})^{1+\alpha}} \right] \geq \frac{(1-r)^{1+\alpha}}{|1 - re^{i(\varphi-\vartheta)}|^{2+2\alpha}} \quad (z = re^{i\varphi})$$

and $d\psi(\vartheta) \geq 0$, from (17) and (18) we shall have

$$|w(z)|^2 \leq \exp\{-\omega(re^{i\varphi})\}; \quad \omega(re^{i\varphi}) = \frac{(1-r)^{1+\alpha}}{\pi} \int_0^{2\pi} \frac{d\psi(\vartheta)}{|1 - re^{i(\varphi-\vartheta)}|^{2+2\alpha}}, \quad (17)$$

and hence $|w(z)| \leq 1$.

Further, since by virtue of (16)

$$|w'(z)| \leq |w(z)| \frac{1+\alpha}{2\pi} \int_0^{2\pi} \frac{d\psi(\vartheta)}{|1 - re^{i(\varphi-\vartheta)}|^{2+\alpha}},$$

then, using Schwarz' s inequality, we obtain

$$|w'(z)|^2 \leq \omega(re^{i\varphi})e^{-\omega(re^{i\varphi})}(1-r)^{-1-\alpha} \int_0^{2\pi} \frac{d\psi(\vartheta)}{|1 - re^{i(\varphi-\vartheta)}|^2}.$$

Finally, observing that $\omega e^{-\omega} \leq e^{-1}$ ($0 \leq \omega < \infty$), by virtue of (17) we arrive at the inequality

$$|w'(z)|^2(1 + |w(z)|^2)^{-2} \leq e^{-1}(1-r)^{-1-\alpha} \int_0^{2\pi} \frac{d\psi(\vartheta)}{|1 - re^{i(\varphi-\vartheta)}|^2}.$$

From this it already follows easily that

$$A(r; w) \leq c_1 r^2 + c_2 (1-r)^{-1-\alpha} \quad (0 \leq r < 1), \quad (18)$$

where $c_1 > 0$ and $c_2 > 0$ do not depend on r . Therefore, for any α' ($-1 < \alpha < \alpha' < 0$),

$$\int_0^1 A(r; w)(1-r)^{\alpha'} dr < +\infty,$$

i.e. $w(z) \in T_{-\alpha'}$.

Proof of Theorem 1 follows immediately from the main theorem A, if one takes into account Theorems 2, 3, 4, Lemma C, and Theorem B.

From Theorem 1 there follows immediately

Theorem 5. The generalized Cauchy-Stieltjes integral $K_\alpha(re^{i\varphi})$ ($-1 < \alpha < 0$) has radial limits (2) everywhere except, possibly, for some set $E \subset [0, 2\pi]$, whose γ -capacity (where $1 + \alpha < \gamma < 1$ —arbitrary) is equal to zero.

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Note: Figure translations are in progress. See original paper for figures.

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