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Abstract

Full Text

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MATHEMATICS

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ON THE QUESTION OF UNIFORM APPROXIMATION OF FUNCTIONS BY POSITIVE LINEAR OPERATORS

(Presented by Academician V. I. Smirnov on 1 IV 1966)

Let a and b , $a < b$, be fixed numbers; let $\omega(h)$, $0 \leq h \leq b - a$, $\omega(0) = 0$, be a nondecreasing function, positive for $0 < h \leq b - a$; let H^ω be the class of functions $f(x)$, $a \leq x \leq b$, such that

$$|f(x_1) - f(x_2)| \leq \omega(|x_1 - x_2|)$$

for x_1 and x_2 from $[a, b]$; let $\sigma(x)$, $a \leq x \leq b$, be a nondecreasing function having an infinite number of points of increase and such that $\sigma(b) - \sigma(a) = 1$; let $\{\varphi_k(x)\}_0^\infty$ be a system of continuous functions orthonormal on $[a, b]$ with integral weight $\sigma(x)$ (see (1), pp. 134, 144), with $\varphi_0(x) \equiv 1$, and let $\lambda = \{\lambda_k\}_0^\infty$ be a collection of real numbers λ_k such that the series

$$K(x, t, \lambda) = \sum_{k=0}^{\infty} \lambda_k \varphi_k(x) \varphi_k(t), \quad a \leq x, t \leq b,$$

converges uniformly and its sum $K(x, t, \lambda) \geq 0$, $a \leq x, t \leq b$.

Let $f(x) \in H^\omega$, and let a_k be the Fourier coefficients of $f(x)$:

$$a_k = \int_a^b f(t) \varphi_k(t) d\sigma(t), \quad k = 0, 1, \dots$$

Introduce the positive linear operator

$$U(x, f, \lambda) = \sum_{k=0}^{\infty} \lambda_k a_k \varphi_k(x) = \int_a^b f(t) K(x, t, \lambda) d\sigma(t).$$

In what follows we put $\lambda_0 = 1$. Then

$$f(x) - U(x, f; \lambda) = \int_a^b [f(x) - f(t)] K(x, t, \lambda) d\sigma(t).$$

Hence, for

$$E(x, \lambda) = \sup_{f \in H^\omega} |f(x) - U(x, f, \lambda)|$$

we obtain the estimate

$$E(x, \lambda) \leq \int_a^b \omega(|x - t|) K(x, t, \lambda) d\sigma(t) = \sum_{k=0}^{\infty} \lambda_k a_k(x) \varphi_k(x),$$

$$a_k(x) = \int_a^b \omega(|x - t|) \varphi_k(t) d\sigma(t). \quad (1)$$

If, for some $x = x_0$, $a \leq x_0 \leq b$, the function $\omega(|x_0 - x|) \in H^\omega$, then in estimate (1) at $x = x_0$ equality holds. In particular, if $\omega(x)$ is a modulus of continuity, then equality in (1) holds for every x , $a \leq x \leq b$.

Example. Let $a = -1$, $b = 1$; $\omega(h) = h$; $\sigma(x) = 1/\pi \arcsin x$; $\varphi_k(x) = \sqrt{2} \cos k \arccos x = \sqrt{2} T_k(x)$, $k = 1, 2, \dots$. In this case

$$K(\cos \theta, \cos \varphi, \lambda) = 1 + 2 \sum_{k=1}^{\infty} \lambda_k \cos k\theta \cos k\varphi =$$

$$= \frac{1}{2} K(\theta - \varphi, \lambda) + \frac{1}{2} K(\theta + \varphi, \lambda),$$

$$K(\theta, \lambda) = 1 + 2 \sum_{k=1}^{\infty} \lambda_k \cos k\theta,$$

and, consequently, in order that the function $K(x, t, \lambda)$ be nonnegative for $-1 \leq x, t \leq 1$, it is necessary and sufficient that the function $K(\theta, \lambda)$ be nonnegative for $0 \leq \theta < \pi$.

Let us find $E(x, \lambda)$ for this case. Putting $x = \cos \theta$, $t = \cos \varphi$, $0 \leq \theta, \varphi \leq \pi$, we have

$$a_0(x) = \frac{1}{\pi} \int_a^b |x - t| \frac{dt}{\sqrt{1 - t^2}} = \frac{1}{\pi} \int_0^\pi |\cos \theta - \cos \varphi| d\varphi =$$

$$= \left(1 - \frac{2\theta}{\pi}\right) \cos \theta + \frac{2}{\pi} \sin \theta,$$

$$a_1(x) = -\frac{1}{\sqrt{2}} \left(1 - \frac{2\theta}{\pi}\right) - \frac{1}{\sqrt{2}\pi} \sin 2\theta,$$

$$a_k(x) = \frac{\sqrt{2}}{\pi} \left[\frac{1}{k(k-1)} \sin(k-1)\theta - \frac{1}{k(k+1)} \sin(k+1)\theta \right], \quad k = 2, 3, \dots,$$

and formula (1) gives

$$E(x, \lambda) = (1 - \lambda_1) \left(1 - \frac{2}{\pi} \theta \right) \cos \theta + \\ + \frac{1}{\pi} \left[2 \left(1 - \frac{\lambda_1}{4} - \sum_{k=2}^{\infty} \frac{\lambda_k}{k^2 - 1} \right) - \sum_{k=1}^{\infty} \frac{\lambda_k - \lambda_{k+1}}{k(k+1)} \sin(2k+1)\theta \right].$$

Returning to the variable x , we finally obtain

$$E(x, \lambda) = (1 - \lambda_1)x \left(1 - \frac{2}{\pi} \arccos x \right) + \\ + \frac{1}{\pi} \sqrt{1 - x^2} \left[2 \left(1 - \frac{\lambda_1}{4} - \sum_{k=2}^{\infty} \frac{\lambda_k}{k^2 - 1} \right) - \sum_{k=1}^{\infty} \frac{\lambda_k - \lambda_{k+1}}{k(k+1)(2k+1)} T'_{2k+1}(x) \right].$$

If here we put $\lambda_k = (n - k)/n$, $k = 1, 2, \dots, n - 1$, $\lambda_k = 0$, $k = n, n + 1$, then we arrive at the result of A. F. Timan² (obtained by him by another method) for the arithmetic means of the Fourier-Chebyshev series.

In conclusion I express my deep gratitude to N. A. Lebedev for useful advice.

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REFERENCES

- ¹ V. L. Goncharov, *Theory of Interpolation and Approximation of Functions*, 1954.
² A. F. Timan, DAN, 77, No. 6, 969 (1951).

Note: Figure translations are in progress. See original paper for figures.

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