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1967

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Abstract

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UDC 517.512.6

MATHEMATICS

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ON THE UNIFORM CONVERGENCE OF FOURIER SERIES IN JACOBI POLYNOMI- ALS

(Presented by Academician M. A. Lavrent'ev, 14 XI 1966)

Let the function $f(x)$ be continuous and have bounded variation on the interval $[-1, 1]$, and let the series

$$\sum_{k=0}^{\infty} c_k \hat{P}_k^{(\alpha, \beta)}(x) \quad (1)$$

be its Fourier series in Jacobi polynomials, orthonormal on the interval $[-1, 1]$ with weight $p(x) = (1-x)^\alpha(1+x)^\beta$. From the equiconvergence theorem of G. Szegő (⁵, p. 254) it follows that for $\alpha > -1$, $\beta > -1$ the series (1) converges uniformly on the interval $[-1 + \mu, 1 - \mu]$, where $0 < \mu < 1$. In the case of Legendre polynomials ($\alpha = \beta = 0$), E. V. Hobson (⁴, p. 325) also established that the series (1) converges at the points $x = \pm 1$; if, moreover, $-1 < \alpha < -1/2$ and $-1 < \beta < -1/2$, then the convergence of the series (1) at these points follows from the results of G. Rau (¹).

In the present paper, the uniform convergence of the series (1) is established on the whole interval $[-1, 1]$ under the conditions $-1 < \alpha < 1/2$, $-1 < \beta < 1/2$, and $|\alpha - \beta| < 1$ (as is known, the case $\alpha = \beta = -1/2$ has been well studied).

Introduce the notation

$$K_n^{(\alpha, \beta)}(t, x) = \sum_{k=0}^n \hat{P}_k^{(\alpha, \beta)}(t) \hat{P}_k^{(\alpha, \beta)}(x). \quad (2)$$

Lemma 1. For the kernel (2) the following formula holds

$$K_n^{(\alpha, \beta)}(t, x) = (1-x^2)K_{n-1}^{(\alpha+1, \beta+1)}(t, x) + \quad (3)$$

$$+ \theta_n^{(\alpha, \beta)} \left[(n + \alpha + \beta + 2) P_n^{(\alpha+1, \beta+1)}(t) P_n^{(\alpha, \beta)}(x) + (n + 1) P_{n-1}^{(\alpha+1, \beta+1)}(t) P_{n+1}^{(\alpha, \beta)}(x) \right],$$

where

$$P_n^{(\alpha, \beta)}(x) = \sqrt{\frac{2^{\alpha+\beta+1} \Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{(2n+\alpha+\beta+1) \Gamma(n+\alpha+\beta+1) \Gamma(n+1)}} \hat{P}_n^{(\alpha, \beta)}(x), \quad (4)$$

$$\theta_n^{(\alpha, \beta)} = O(1), \quad K_{-1}^{(\alpha+1, \beta+1)}(t, x) \equiv 0, \quad P_{-n}^{(\alpha+1, \beta+1)}(t) \equiv 0.$$

Formula (3) is a consequence of known relations for Jacobi polynomials ⁽⁴⁾, contained in ⁽⁵⁾, Ch. IV.

Put

$$\Phi_n^{(\alpha, \beta)}(a, b; x) = \int_a^b p(t) K_n^{(\alpha, \beta)}(t, x) dt, \quad -1 \leq a < b \leq 1, \quad x \in [-1, 1], \quad (5)$$

and by $\Phi_n^{(\alpha, \beta)}(E; x)$ we shall denote the same integral, taken over a set $E \subset [-1, 1]$. In view of the validity of the formula ⁽⁵⁾, p. 71)

$$P_n^{(\alpha, \beta)}(x) = (-1)^n P_n^{(\beta, \alpha)}(-x) \quad (6)$$

it is sufficient to consider the case when $x \in [0, 1]$.

Lemma 2. Let $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$, $\beta > -1$, and $x \in [0, 1]$. Then, provided the condition $\alpha - \beta \leq 1$ is satisfied, there exists a constant C_1 , depending only on α and β , such that for all n the inequality

$$|\Phi_n^{(\alpha, \beta)}(a, b; x)| \leq C_1, \quad -1 \leq a < b \leq 1. \quad (7)$$

holds.

Proof. Let $E_1 = \{t; |t - x| < \delta\}$, where $0 < \delta < 1$. We first consider the case when $[a, b] \cap E_1 = \emptyset$. Using formula ((5), p. 83)

$$K_n^{(\alpha, \beta)}(t, x) = l_n^{(\alpha, \beta)} \frac{P_{n+1}^{(\alpha, \beta)}(t) P_n^{(\alpha, \beta)}(x) - P_n^{(\alpha, \beta)}(t) P_{n+1}^{(\alpha, \beta)}(x)}{t - x}, \quad (8)$$

where $l_n^{(\alpha, \beta)} = O(n)$, and the second mean-value theorem, from (5) we obtain

$$|\Phi_n^{(\alpha, \beta)}(a, b; x)| \leq C_2 \delta^{-1} n \int_{a_1}^{b_1} p(t) [P_{n+1}^{(\alpha, \beta)}(t) P_n^{(\alpha, \beta)}(x) - P_n^{(\alpha, \beta)}(t) P_{n+1}^{(\alpha, \beta)}(x)] dt. \quad (9)$$

where $[a_1, b_1] \subset [a, b]$. For the polynomials (4) the following relation holds ((5), p. 175)

$$\max_{x \in [0,1]} |P_n^{(\alpha, \beta)}(x)| = |P_n^{(\alpha, \beta)}(1)| = \frac{\Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(n + 1)}, \quad \alpha \geq -\frac{1}{2}, \quad (10)$$

and the inequality

$$(1-x)^{\alpha/2+1/4}(1+x)^{\beta/2+1/4} |P_n^{(\alpha, \beta)}(x)| \leq \frac{C_3}{\sqrt{n}},$$

$$\min(\alpha, \beta) = \rho \geq -\frac{1}{2}, \quad x \in [-1, 1], \quad (11)$$

which is a consequence of formula 7.32.5 ((5), p. 176). If $\beta \geq -\frac{1}{2}$, then by means of (10), (11), and formula ((5), p. 107)

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha, \beta)}(x) = -\frac{1}{2n} \frac{d}{dx} \left\{ (1-x)^{\alpha+1}(1+x)^{\beta+1} P_{n-1}^{(\alpha+1, \beta+1)}(x) \right\} \quad (12)$$

from (9) we find

$$|\Phi_n^{(\alpha, \beta)}(a, b; x)| \leq C_4 \delta^{-1} n^{\alpha-1/2}. \quad (13)$$

If $-1 < \beta < -\frac{1}{2}$, then instead of (11) we use the inequality of G. Szegő ⁽²⁾

$$(1+x)^\lambda |P_n^{(\alpha, \beta)}(x)| \leq C_5 n^{\beta-2\lambda}, \quad (14)$$

where $\beta > -\frac{1}{2}$, $0 \leq \lambda < \beta/2 + \frac{1}{4}$, $x \in [-1, 1 - \mu]$, and obtain

$$|\Phi_n^{(\alpha, \beta)}(a, b; x)| \leq C_6 \delta^{-1} n^{\alpha-\beta-1}. \quad (15)$$

Now let $[a, b] \cap E_1 = E_2$ and $E_2 \neq \emptyset$, while $E_3 = [a, b] \setminus E_2$. For $\Phi_n^{(\alpha, \beta)}(E_3; x)$ estimates analogous to (13) or (15) hold. To obtain an estimate for $\Phi_n^{(\alpha, \beta)}(E_2; x)$, we make the substitution $t = \cos \theta$. To the set E_2 there corresponds a set $D \subset [0, \pi]$, which we split into parts: $D_1 = \{\theta; |\theta - \theta_0| \leq 1/n\}$ and $D_2 = D \setminus D_1$; here $\theta_0 = \arccos x$.

a) Let $D_1 \neq \emptyset$. From (10) and (11) we obtain

$$|P_n^{(\alpha, \beta)}(\cos \theta)| \leq C_7 \theta^{-\alpha-1/2} [1 - \varphi_n(\theta)] n^{-1/2} + C_8 n^\alpha \varphi_n(\theta), \quad (16)$$

$$\alpha \geq -\frac{1}{2}, \quad 0 \leq \theta \leq \pi/2, \quad \varphi_n(\theta) = \begin{cases} 1, & 0 \leq \theta \leq n^{-1}, \\ 0, & n^{-1} < \theta \leq \pi/2. \end{cases}$$

Taking into account (4), (5), and (16), we arrive at the inequality

$$\begin{aligned} |\Phi_n^{(\alpha, \beta)}(D_1; \cos \theta_0)| &\leq C_9 \left[\int_{D_1} \theta^{2\alpha+1} d\theta \right]^{1/2} \left[\sum_{k=0}^n [\widehat{P}_k^{(\alpha, \beta)}(\cos \theta_0)]^2 \right]^{1/2} \leq \\ &\leq C_{10} \left[\int_{D_1} \sum_{k=0}^n \left\{ [1 - \varphi_k(\theta_0)] \left(\frac{\theta}{\theta_0} \right)^{\alpha+1/2} + (k\theta)^{\alpha+1/2} \varphi_k(\theta_0) \right\}^2 d\theta \right]^{1/2} \leq C_{11}. \end{aligned} \quad (17)$$

b) Let, next, $D_2^{(1)} = D_2 \cap (\theta_0 + n^{-1}, \arccos(x - \delta))$. Then, using formula (3) and introducing for the integrals obtained the notation I_1, I_2, I_3 , we have

$$\begin{aligned} |\Phi_n^{(\alpha, \beta)}(D_2^{(1)}; \cos \theta_0)| &\leq C_{12} \left\{ \left| \int_{D_2^{(1)}} \left(\sin \frac{\theta}{2} \right)^{2\alpha+1} \left(\cos \frac{\theta}{2} \right)^{2\beta+1} \sin^2 \theta_0 K_{n-1}^{(\alpha+1, \beta+1)}(\cos \theta, \cos \theta_0) d\theta \right| \right. \\ &\quad + \left| \int_{D_2^{(1)}} \left(\sin \frac{\theta}{2} \right)^{2\alpha+1} \left(\cos \frac{\theta}{2} \right)^{2\beta+1} n P_n^{(\alpha, \beta)}(\cos \theta_0) P_n^{(\alpha+1, \beta+1)}(\cos \theta) d\theta \right| \\ &\quad + \left| \int_{D_2^{(1)}} \left(\sin \frac{\theta}{2} \right)^{2\alpha+1} \left(\cos \frac{\theta}{2} \right)^{2\beta+1} n P_{n+1}^{(\alpha, \beta)}(\cos \theta_0) P_{n-1}^{(\alpha+1, \beta+1)}(\cos \theta) d\theta \right| \Big\} \\ &= C_{12} \{I_1 + I_2 + I_3\}. \end{aligned} \quad (18)$$

To estimate I_1 we use inequality (11), the asymptotic formula ((5), p. 205)

$$P_n^{(\alpha, \beta)}(\cos \theta) = n^{-1/2} k(\theta) \{ \cos(N\theta + \gamma) + (n \sin \theta)^{-1} O(1) \}, \quad (19)$$

where

$$k(\theta) = \pi^{-1/2} \left(\sin \frac{\theta}{2} \right)^{-\alpha-1/2} \left(\cos \frac{\theta}{2} \right)^{-\beta-1/2}, \quad N = n + \frac{\alpha + \beta + 1}{2},$$

$$\gamma = -(\alpha + 1/2)\pi/2, \quad \alpha > -1, \quad \beta > -1, \quad cn^{-1} \leq \theta \leq \pi - cn^{-1},$$

$c > 0$ is a fixed number, and the identity ((5), p. 83)

$$\begin{aligned} & \frac{2n + \alpha + \beta + 2}{2}(1-x)P_n^{(\alpha+1, \beta)}(x) = \\ & = (n + \alpha + 1)P_n^{(\alpha, \beta)}(x) - (n + 1)P_{n+1}^{(\alpha, \beta)}(x). \end{aligned} \quad (20)$$

After transformations from (18) we obtain

$$\begin{aligned} |I_1| \leq C_{13} \left\{ \left| \int_{D_2^{(1)}} \left(\frac{\theta_0}{\sin \frac{\theta}{2}} \right)^{1/2-\alpha} \frac{\theta_0 [\cos(N_1\theta + \gamma_1) + (n \sin \theta)^{-1}]}{(\cos \frac{\theta}{2})^{1/2-\beta} \sin \frac{\theta-\theta_0}{2} \sin \frac{\theta+\theta_0}{2}} d\theta \right| \right. \\ \left. + \int_{D_2^{(1)}} \left(\frac{\theta_0}{\sin \frac{\theta}{2}} \right)^{1/2-\alpha} \frac{\sin \frac{\theta}{2} [\cos(N_2\theta + \gamma_2) + (n \sin \theta)^{-1}]}{(\cos \frac{\theta}{2})^{1/2-\beta} \sin \frac{\theta-\theta_0}{2} \sin \frac{\theta+\theta_0}{2}} d\theta \right\}, \end{aligned}$$

whence, after applying the second mean-value theorem, the estimate $|I_1| \leq C_{14}$ follows. With the help of (10) and (19) it is also not difficult to establish the boundedness of the integrals I_2 and I_3 in (18), which leads to the inequality

$$|\Phi_n^{(\alpha, \beta)}(D_2^{(1)}; \cos \theta_0)| \leq C_{15}.$$

- c) Let, finally, $D_2^{(2)} = D_2 \setminus D_2^{(1)}$ and $D_2^{(2)} \neq \emptyset$. If $1/n \in D_2^{(2)}$, then, arguing as in case a), we find the inequality $|\Phi_n^{(\alpha, \beta)}(\cos 1/n, b; x)| \leq C_{16}$. The further arguments are carried out with the aid of relations (8), (11), (12), (19), and (20), after which we obtain the estimate

$$|\Phi_n^{(\alpha, \beta)}(D_2^{(2)}; \cos \theta_0)| \leq C_{17},$$

and Lemma 2 is proved.

Let us note that inequality (7) in the case when $\alpha = \beta = 1/2$ (Chebyshev polynomials of the second kind) is not difficult to verify directly.

Lemma 3. *If $-1 < \alpha < -1/2$ and $\beta > -1$, then there exists a constant C_{18} , depending on α and β , such that for all $x \in [0, 1]$ and all n the inequality holds*

$$|\Phi_n^{(\alpha, \beta)}(a, b; x)| \leq C_{18}, \quad -1 \leq a < b \leq 1. \quad (21)$$

The proof is essentially the same as the preceding one. To estimate $\Phi_n^{(\alpha, \beta)}(E_2; x)$, the relation

$$\begin{aligned}\Phi_n^{(\alpha, \beta)}(E_2; x) &= \Phi_{n-1}^{(\alpha+1, \beta+1)}(E_2; x) + \\ &+ \theta_n^{(\alpha, \beta)} \int_{E_2} p(t) [(n + \alpha + \beta + 2) P_n^{(\alpha+1, \beta+1)}(x) P_n^{(\alpha, \beta)}(t) + \\ &+ (n + 1) P_{n-1}^{(\alpha+1, \beta+1)}(x) P_{n+1}^{(\alpha, \beta)}(t)] dt,\end{aligned}$$

which is a consequence of (3), is additionally used.

Thus, under the conditions $-1 < \alpha \leq 1/2$, $-1 < \beta \leq 1/2$, and $|\alpha - \beta| \leq 1$, Lemmas 2, 3 and relation (7) imply the inequality

$$|\Phi_n^{(\alpha, \beta)}(a, b; x)| \leq C_{19}, \quad -1 \leq a < b \leq 1, \quad -1 \leq x \leq 1. \quad (22)$$

Theorem. *If $-1 < \alpha < 1/2$, $-1 < \beta < 1/2$, and $|\alpha - \beta| < 1$, then every continuous function $f(x)$ of bounded variation on the interval $[-1, 1]$ is expanded in a Fourier-Jacobi series converging uniformly on this interval.*

Proof. It is enough to consider the case where $f(x)$ is continuous and nondecreasing on the interval $[-1, 1]$. As is known (3), the remainder of the series (1) may be represented in the form

$$R_n^{(\alpha, \beta)}(f; x) = \int_{-1}^1 [f(x) - f(t)] p(t) K_n^{(\alpha, \beta)}(t, x) dt,$$

and the validity of the assertion of the theorem follows from relations (8), (22), (11) or (14).

In conclusion we give two monotone continuous functions with divergent Fourier-Jacobi series under some conditions on α and β . Let $f_1(x) = \sqrt{1+x}$ for $x \in [-1, 0]$, and $f_1(x) \equiv 1$ for $x \in (0, 1]$. Then for $\alpha = \beta = 1/2$ we obtain

$$R_n^{(1/2, 1/2)}(f; 1) = -(-1)^{[(n+1)/2]} / \pi.$$

If, however, we put

$$f_2(x) = (1+x)^{(\alpha-\beta-1)/2},$$

where $\alpha - \beta > 1$, then, using a result of G. Szegő ((5), p. 264), we find that the Fourier-Jacobi series of this function also diverges at the point $x = 1$.

The author expresses sincere gratitude to P. K. Suetin, under whose supervision this work was carried out.

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Received
20 IX 1966

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Note: Figure translations are in progress. See original paper for figures.

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