

ON AN INHOMOGENEOUS SINGULAR CAUCHY PROBLEM

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Abstract

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MATHEMATICS

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ON AN INHOMOGENEOUS SINGULAR CAUCHY PROBLEM

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Consider, in the $(n+1)$ -dimensional domain $\Omega[-\infty < x < \infty, 0 \leq s < \infty, x = (x_1, \dots, x_n)]$, the solution $u(x, s; a, b, c)$ of the inhomogeneous singular Cauchy problem

$$Xu = u_{ss} + \frac{a}{s}u_s + \left(b^2 + \frac{c}{s^2}\right)u - \frac{c}{s^2}\tau(x), \quad u(x, 0) = \tau(x), \quad u_s(x, 0) = 0, \quad (1)$$

where $a = 2\beta \geq 0$; $b, c \leq \nu^2$; $\nu = \beta - \frac{1}{2}$ are constants; $\tau(x) \in C^\infty$ is given on the entire hyperplane $\mathfrak{S}_n(-\infty < x_k < \infty; k = 1, 2, \dots, n)$; X is a linear differential operator independent of s , acting with respect to x , and $X[0] = 0$. Comparing $u(x, s)$ with $z(x, s; a, b) = u(x, s; a, b, 0)$, we arrive at the following results:

1. Let $2^{2m}m!(\nu+1)_mg_m(\nu) = (-1)^m$, $b_0 = \sqrt{b_1^2 - b_2^2}$, $4\sigma = b_0^2s^2$. Then

$$u(x, \lambda s; a_2, b_2, c) = \sum_{m=0}^{\infty} A_m s^{2m} (X - b_1^2)^m z(x, s; a_1 + 2m, b_1), \quad (2a)$$

$$A_m = g_m(\nu_1) \sum_{n=0}^m \frac{(-m)_n (\nu_1 + 1)_n \lambda^{2n}}{(p_2 + 1)_n (q_2 + 1)_n} {}_1F_2(n+1; p_2+n+1, q_2+n+1; \lambda^2 \sigma). \quad (2b)$$

Here $\lambda = \text{const}$; p_k and q_k are the roots of the equation $\rho^2 - \nu_k \rho + c/4 = 0$.

$$u(x, \lambda s; a_2, b_2, c) = \sum_{m=0}^{\infty} \bar{A}_m s^{2m} (X - b_1^2)^m z(x, s; a_1 + 4m, b_1), \quad (3a)$$

$$\bar{A}_m = \bar{g}_m(\nu_1) \sum_{n=0}^m \frac{(-m)_n (\nu_1 + m)_n \lambda^{2n}}{(p_2 + 1)_n (q_2 + 1)_n} {}_1F_2(n+1; p_2+n+1, q_2+n+1; \lambda^2 \sigma), \quad (3b)$$

where $2^{2m}m!(\nu+m)_m\bar{g}_m(\nu)=(-1)^m$. With the aid of the equality

$$s^{2m}(X-b^2)^m z(x, s; a+4m, b) = \sum_{n=0}^m \gamma_n z(x, s; a+2n, b), \quad (4)$$

$$n!(\nu+1)_n \gamma_n = 2^{2m}(-m)_n(\nu+1)_{2m}(\nu+m)_n \quad (m, n = 0, 1, 2, \dots),$$

one may pass from (3a) to an expansion in the family $\{z(a+2n)\}$.

For example, when $a_2 = a_1 = a$, $b_2 = b_1 = b$, substituting (4) into (3a), we find:

$$u(x, s; a, b, c) = \sum_{n=0}^{\infty} \frac{pq\Gamma(p+n)\Gamma(q+n)}{n!\Gamma(\nu+n+1)} z(x, s; a+2n, b). \quad (5)$$

Under the conditions $a_2 > a_1 \geq 0$, $\beta_0 = \beta_2 - \beta_1$, $\Gamma(\beta_0)\Gamma(\nu_1+1)\mu =$

$$= 2\Gamma(p_2+1)\Gamma(q_2+1),$$

(2), (3), and (5) generates the connection formula

$$u(x, s; a_2, b_2, c) = \int_0^1 \xi^{a_1}(1-\xi^2)^{\beta_0-1} Q(\xi, s) z(x, \xi s; a_1, b_1) d\xi, \quad (6a)$$

$$Q(\xi, s) = \mu \Xi_2[p_2, q_2, \beta_0; 1-\xi^2, \sigma(1-\xi^2)]. \quad (6b)$$

When $b_2 = b_1 = b$, (2b) and (3b) give

$$\begin{aligned} A_m &= g_m(\nu_1) {}_3F_2(-m, \nu_1+1, 1; p_2+1, q_2+1; \lambda^2), \\ \bar{A}_m &= \bar{g}_m(\nu_1) {}_3F_2(-m, \nu_1+m, 1; p_2+1, q_2+1; \lambda^2), \end{aligned} \quad (7)$$

and (6b) becomes $Q_1 = \mu F(p_2, q_2; \beta_0; 1-\xi^2)$. Conversely, in the case $c = 0$ ($p = 0$, $q = \nu$), where $\Gamma(\beta_0)\Gamma(\nu_1+1)\mu_2 = 2\Gamma(\nu_2+1)$,

$$A_m = g_m(\nu_1) \Xi_2(-m, \nu_1+1; \nu_2+1; \lambda^2, -\lambda^2\sigma), \quad (8a)$$

$$\bar{A}_m = \bar{g}_m(\nu_1) \Xi_2(-m, \nu_1+m; \nu_2+1; \lambda^2, -\lambda^2\sigma), \quad (8b)$$

the series (6b) reduces to $Q_2 = \mu_2 \bar{I}_{\beta_0-1}(b_0 s \sqrt{1-\xi^2})$ (1). For $a_2 = a_1 = a$, $c = 0$, from (3b) and (8b) there arises the addition theorem

$$z(x, s; a, b_2) = \sum_{m=0}^{\infty} \bar{A}_m s^{2m} (X-b_1^2)^m z(x, s; a+4m, b_1), \quad (9a)$$

$$(\nu+1)_{2m} \bar{A}_m = \bar{g}_m(\nu) \sigma^m \bar{I}_{\nu+2m}(b_0, s), \quad (9b)$$

inverting which, we arrive at the equality

$$\bar{I}_\nu(b_0 s) z(x, s; a, b_1) = \sum_{n=0}^{\infty} B_n (b_0 s^2)^{2n} (X - b_1^2)^n z(x, s; a + 4n, b_2), \quad (10)$$

where $(\nu + 1)_{2n} 2^{2n} B_n = (-1)^n g_n(\nu)$. It is also worth noting the case $p_2 = \beta_0$, $q_2 = \nu_1$ ($a_1 = 2q_2 + 1$), when $A_m = g_m(q) F(-m, 1, p + 1; \lambda^2)$,

$$u(x, \lambda s; a, b, c) = \sum_{m=0}^{\infty} A_m s^{2m} (X - b^2)^m z[x, s; 2(q + m) + 1, b], \quad (11a)$$

$$u(x, s; a, b, c) = p \int_0^1 (1 - \eta)^{p-1} z(x, s\sqrt{\eta}; 2q + 1, b) d\eta \quad (p > 0). \quad (11b)$$

2. Let us replace in (1) s, a, c by $2\sqrt{\varepsilon s}$, $2\varepsilon - 1$, $4c\varepsilon$ and pass to the limit as $\varepsilon \rightarrow \infty$. Then (1) is reduced to the Cauchy problem

$$Xv = v_s + \left(b^2 + \frac{c}{s}\right) v - \frac{c}{s} \tau(x), \quad v(x, 0) = \tau(x), \quad (12a)$$

$$v(x, s; b, c) = \lim_{\varepsilon \rightarrow \infty} u(x, 2\sqrt{\varepsilon s}; 2\varepsilon - 1, b, 4c\varepsilon). \quad (12b)$$

Conversely, conflating u and z according to the rule

$$\lim_{\varepsilon \rightarrow \infty} u(x, 2\sqrt{\varepsilon s}; 2\varepsilon - 1, b, c) = \lim_{\varepsilon \rightarrow \infty} z(x, 2\sqrt{\varepsilon s}; 2\varepsilon - 1, b) = w(x, s; b), \quad (13)$$

we arrive at the problem, regular with respect to s , with initial condition

$$Xw = w_s + b^2 w, \quad w(x, 0) = \tau(x), \quad (x, s) \in \Omega. \quad (14)$$

For $a_1 = 2\varepsilon - 1$, $a_2 = a$, $s = 2\sqrt{\varepsilon s_1}$, $\lambda = \sqrt{\lambda_1/\varepsilon}$, $\varepsilon \rightarrow \infty$, (7) and (8a) give

$$u(x, 2\sqrt{\lambda_1 s_1}; a, b, c) = \sum_{m=0}^{\infty} A_m^{(1)} s_1^m (X - b^2)^m w(x, s_1; b), \quad (15a)$$

$$z(x, 2\sqrt{\lambda_1 s_1}; a, b_2) = \sum_{m=0}^{\infty} \bar{A}_m^{(1)} s_1^m (X - b^2)^m w(x, s_1; b_1), \quad (15b)$$

where

$$m! A_m^{(1)} = (-1)^m {}_2F_2(-m, 1; p + 1, q + 1; \lambda_1),$$

$\bar{A}_m^{(1)}$ is a Humbert function

$$\bar{A}_m^{(1)} = \frac{(-1)^m}{m!} \Phi_3(-m, \nu + 1; \lambda_1, b_0^2 \lambda_1 s_1). \quad (15c)$$

Putting in (2) and (3) $a_1 = a$, $a_2 = 2\varepsilon - 1$, $b_1 = b_2 = b$, $c = 4\varepsilon c_1$, $\lambda = 2\lambda_1\sqrt{\varepsilon}$, $\varepsilon \rightarrow \infty$, we obtain

$$v(x, \lambda_1^2 s^2; b, c_1) = \sum_{m=0}^{\infty} A_m^{(2)} s^{2m} (X - b^2)^m z(x, s; a + 2m, b), \quad (16a)$$

$$v(x, \lambda_1^2 s^2; b, c_1) = \sum_{m=0}^{\infty} \bar{A}_m^{(2)} s^{2m} (X - b^2)^m z(x, s; a + 4m, b), \quad (16b)$$

$$A_m^{(2)} = \bar{g}_{m3} F_1(-m, \nu + 1, 1; c_1 + 1; 4\lambda_1^2),$$

$$\bar{A}_m^{(2)} = \bar{g}_{m3} F_1(-m, \nu + m, 1; c_1 + 1; 4\lambda_1^2).$$

Let us also note a relation, similar to (15b), with the Humbert function Φ_1 :

$$v(x, \lambda s; b_2, c) = \sum_{m=0}^{\infty} A_m s^m (X - b_1^2)^m w(x, s; b_1), \quad (17a)$$

$$m! A_m = (-1)^m \Phi_1(1, -m; c + 1; \lambda, b_0^2 \lambda s). \quad (17a)$$

3. When $b_2 = b_1 = b$, (8b) reduces to Jacobi polynomials $\bar{A}_m = \bar{g}_m(\nu_1) G_m(\nu_1, \nu_2 + 1; \lambda^2)$, and therefore, if $z(x, \lambda s; a_2, b) = f(\lambda)$, then

$$s^{2m} (X - b^2)^m z(x, s; a_1 + 4m, b) = \delta_m \int_0^1 \lambda^{\alpha_2} (1 - \lambda^2)^{-\beta_0 - 1} \bar{A}_m(\lambda) f(\lambda) d\lambda,$$

$$\Gamma(m - \beta_0) \Gamma(\nu_2 + 1) \bar{g}_m(\nu_1) \delta_m = (-1)^m 2^{2m+1} (\nu_2 + 1)_m \Gamma(\nu_1 + 2m + 1). \quad (18)$$

For $b_2 = b_1 = b$, it follows from (15c) that

$(\nu + 1)_m \bar{A}_m^{(1)} = (-1)^m L_m^{(\nu)}(\lambda_1)$, and the Fourier coefficients of the function $z(x, 2\sqrt{\lambda_1} s; a, b) = f(\lambda_1)$ are

$$s_1^m (X - b^2)^m w(x, s_1; b) = \frac{(-1)^m m!}{\Gamma(\nu + 1)} \int_0^\infty \lambda_1^\nu e^{-\lambda_1} L_m^{(\nu)}(\lambda_1) f(\lambda_1) d\lambda_1. \quad (19)$$

Replacing the bases of the expansions (15), (16b), (17a) by their integral representations (18) and (19), we arrive at analogous relations for u and v . For example, (17a) and (19) give

$$v(x, s; b_2, c) = \int_0^\infty \lambda^{\nu/2} R(\lambda, s) z(x, 2\sqrt{\lambda}s; a, b_1) d\lambda, \quad (20a)$$

$$\Gamma(\nu + 1)R = \int_0^\infty \xi^{\nu/2} e^{-\xi} J_\nu(2\sqrt{\lambda\xi}) \Phi_2(1, c, c + 1; b_0^2 s, \xi) d\xi. \quad (20b)$$

Let us also note the relation, similar to (19) and (20a),

$$v(x, s; b, p) = \frac{1}{\Gamma(q + 1)} \int_0^\infty \lambda^q e^{-\lambda} u(x, 2\sqrt{\lambda}s; a, b, c) d\lambda \quad (q > -1), \quad (21)$$

and, in the case $c_2 > c_1 > -1$, the addition theorem:

$$v(x, s; b, c_2) = \frac{\Gamma(c_2 + 1)}{\Gamma(c_1 + 1)\Gamma(c_2 - c_1)} \int_0^1 \xi^{c_1} (1 - \xi)^{c_2 - c_1 - 1} v(x, \xi s; b, c_1) d\xi. \quad (22)$$

If $X = \partial/\partial x_1 + \dots + \partial/\partial x_n$, then

$w = e^{-b^2 s} \tau(x_1 + s, \dots, x_n + s)$, and here (15), (17), and (22) (for $c_1 = 0$) give the solutions u, z, v , while (16) (for $c_1 = 0$), (19), and (20) (for $c = 0$) invert the corresponding resolving operators with respect to $\tau(x_1 + s, \dots, x_n + s)$. Suppose, further, that $a = n - 1$, $b = 0$, X is the Laplacian

$$X[z] = \Delta_x[z] = \sum_{i=1}^n z_{x_i x_i}.$$

Then $z(x, s;$

$n - 1, 0) = M[x, s; \tau(x)]$. This means that (6a) in the case $a_1 = n - 1$, $b_1 = 0$, $a_2 > n - 1$, $X = \Delta_x$, and also (20a) for $a = n - 1$, $b_1 = 0$, give solutions u and v of problems (1), (12a). For example, here (6), when $n = 1$, take the form

$$u(x, s; a, b, c) = \int_{-1}^1 (1 - \xi^2)^{\beta-1} Q_0(\xi, s) \tau(x + \xi s) d\xi, \quad (23a)$$

$$Q_0(\xi, s) = \mu_0 \Xi_2[p, q, \beta; 1 - \xi^2, -\sigma_0(1 - \xi^2)], \quad (23b)$$

where $\sqrt{\pi} \Gamma(\beta) \mu_0 = \Gamma(p + 1) \Gamma(q + 1)$, $4\sigma_0 = b^2 s^2$, and from (20a) it follows that

$$v(x, s; 0, c) = \frac{\Gamma(c + 1)}{\sqrt{\pi}} \int_{-\infty}^\infty e^{-\xi} \Psi(c, 1/2; \xi^2) \tau(x + 2\xi\sqrt{s}) d\xi. \quad (24)$$

Analogous assumptions reduce (2a), (3a), (8), and (15b) to expansions of the means $M(x, s; \tau)$ in the basic series over the systems $\{z(a + 2m)\}$, $\{z(a + 4m)\}$, $\{(X - b^2)^m w\}$.

Finally, let $b = 0$, $Xz = \mathfrak{B}_x[z] = z_{xx} + \frac{2a}{x}z_x$. Then

$$z = \gamma \int_{-1}^1 \tau(x + \xi s)(1 - \xi^2)^{\beta-1}(1 + \xi t)^\alpha F(\alpha, 1 - \alpha, \beta; \omega) d\xi, \quad (25)$$

$$\sqrt{\pi}\Gamma(\beta)\gamma = \Gamma(\nu + 1), \quad t = -s/x, \quad 4(1 + \xi t)\omega = t^2(\xi^2 - 1).$$

Substituting (25) into (6a) ($b_1 = 0$), (11b), (19), and (20a) ($b = 0$), we obtain u, v, w for $X = \mathfrak{B}_x$.

4. To power data $\tau(x) = x^\alpha$ ($\alpha = \text{const}$) for $X = \partial/\partial x$ there correspond the solutions

$$u_1 = x^\alpha {}_2F_2(-\alpha, 1; p + 1, q + 1; -t_1), \quad w_1 = e^{-\sigma_1}(x + s)^\alpha,$$

$$z_1 = x^\alpha \Phi_3(-\alpha, \nu + 1; -t_1, -\sigma_0), \quad v_1 = x^\alpha \Phi_1(-\alpha, 1, c + 1; -t, -\sigma_1),$$

where $t_1 = s^2/4x$, $\sigma_1 = b^2s$. If, however, $X = \partial^2/\partial x^2$, then

$$z_2 = x^\alpha \Xi_2\left(-\frac{\alpha}{2}, \frac{1-\alpha}{2}, \nu + 1; t^2, -\sigma_0\right),$$

$$u_2 = x^\alpha {}_3F_2\left(-\frac{\alpha}{2}, \frac{1-\alpha}{2}, 1, p + 1, q + 1; t^2\right).$$

Finally, confluenting z_2 and u_2 according to the rules (12b), (13), we obtain

$$w_2 = x^\alpha e^{-\sigma_1} \times {}_2F_0\left(-\frac{\alpha}{2}, \frac{1-\alpha}{2}; t_2\right),$$

$$v_2 = x^\alpha {}_3F_1\left(-\frac{\alpha}{2}, \frac{1-\alpha}{2}, 1, c + 1; t_2\right),$$

where $t_2 = 4s/x^2$.

In the case $\mathfrak{B}_x z = z_{ss} + \frac{a}{s}z_s$, the solutions z and $\bar{z}$ of Tricomi's problems from (2) are written in quadratures with the aid of the Green-Hadamard functions $H(x, s; x_0, s_0)$, $\bar{H}(x, s; x_0, s_0)$, which here have the form:

$$H = \chi t^{1-a} \left(\frac{2s_0}{R} \right)^a \left(\frac{x}{x_0} \right)^\alpha H_2(1-\beta, 1-\beta, \alpha, 1-\alpha, 2-a; t, \rho),$$

$$\overline{H} = \overline{\chi} \left(\frac{2s_0}{R} \right)^a \left(\frac{x}{x_0} \right)^\alpha H_2(\beta, \beta, \alpha, 1-\alpha, a; t, \rho),$$

where $tR^2 = 4ss_0$, $4xx_0\rho = R^2$, $R = \sqrt{(x-x_0)^2 - (s-s_0)^2}$; χ and $\overline{\chi}$ are indicated in (2).

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Note: Figure translations are in progress. See original paper for figures.

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