

ON AN ANALOGUE OF THE ANDRONOV-WITT THEOREM

B. P. Demidovich

1967

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196701.55993>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.919

MATHEMATICS

B. P. DEMIDOVICH

ON AN ANALOGUE OF THE ANDRONOV-WITT THEOREM

(Presented by Academician A. N. Kolmogorov, 22 XII 1966)

1°. In this paper sufficient conditions are obtained for the asymptotic orbital stability of a completely bounded solution of an autonomous system of ordinary differential equations, analogous to the well-known Andronov-Witt theorem ⁽¹⁾ on the stability of a periodic solution.

2°. Consider the real linear system

$$dx/dt = A(t)x, \quad (1)$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \equiv \text{colon}(x_1, \dots, x_n),$$

$$A(t) \in C(I_t^+) \quad (I_t^+ = (t_0, +\infty)) \text{ is an } n \times n\text{-matrix; } \sup_t \|A(t)\| < \infty.$$

By

$$\chi[\mathbf{x}(t)] = \lim_{t \rightarrow \infty} \left\{ \frac{1}{t} \ln \|\mathbf{x}(t)\| \right\}$$

we shall denote the **characteristic exponent** of the solution $\mathbf{x}(t)$.

Definition 1. A bounded solution $\mathbf{x}(t)$ of system (1) will be called **standard** if

$$0 < \inf_t \|\mathbf{x}(t)\| \leq \sup_t \|\mathbf{x}(t)\| < \infty.$$

Definition 2. A bounded solution $\mathbf{x}(t)$ of system (1) will be called **completely bounded** if: 1) it is standard and 2) the adjoint system

$$d\vec{\xi}/dt = A^T(t)\vec{\xi} \quad (2)$$

($A^T(t)$ is the transposed matrix of $A(t)$) has a bounded solution $\vec{\xi}(t)$, with

$$\mathbf{x}^T(t)\vec{\xi}(t) = \text{const} \neq 0.$$

Definition 3. A square matrix $L(t)$ will be called a **Lyapunov matrix** on the set $Z \subset I_t^+$ if, for $t \in Z$, the following conditions are satisfied:

- a) $L(t) \in C^{(1)}$;
- b) $\sup_t \|L(t)\| < \infty$, $\sup_t \|\dot{L}(t)\| < \infty$;
- c) $\inf_t |\det L(t)| > 0$.

Lemma. Let system (1) be regular in the sense of Lyapunov ⁽²⁾, and let its normal fundamental matrix $X(t) = (x_{ij}(t))$ consist of solutions

$$\mathbf{x}^{(j)}(t) = \text{colon}[x_{1j}(t), \dots, x_{nj}(t)] \quad (j = 1, \dots, n),$$

of which one, $\mathbf{x}^{(1)}(t)$, is completely bounded, while all the remaining $\mathbf{x}^{(j)}(t)$ ($j > 1$) have negative characteristic exponents.

Then the half-plane $\{t_0 \leq t < +\infty, t_0 \leq \tau < +\infty\}$ can be represented as a finite sum of open sets $O_{pq} = \{t, \tau : t \in O_p, \tau \in O_q\}$, on each of which the Cauchy matrix

$$K(t, \tau) = X(t)X^{-1}(\tau)$$

admits the representation

$$K(t, \tau) = U_p(t) \text{diag}[E_1, Y_{pq}(t, \tau)] V_q^T(\tau), \quad (3)$$

where E_1 is the identity matrix of order 1, $U_p(t)$ and $V_q(\tau)$ are Lyapunov matrices on O_p and O_q , respectively, $Y_{pq}(t, \tau) \in C^{(1)}(O_{pq})$, and

$$\|Y_{pq}(t, \tau)\| \leq ce^{-\alpha(t-\tau)}e^{\varepsilon t} \quad (4)$$

for $t_0 \leq \tau \leq t$, where $\varepsilon > 0$ is arbitrary,

$$-\alpha = \max_{j>1} \chi[\mathbf{x}^{(j)}(t)] < 0$$

and $c = c(\varepsilon, t_0)$ is a positive constant.

3°. Let the real autonomous system

$$dy/dt = \mathbf{f}(\mathbf{y}), \quad (5)$$

where $\mathbf{f}(\mathbf{y}) \in C^{(2)}(R^n)$, $n \geq 2$, admit a solution $\eta = \eta(t)$ bounded on $I_t = (-\infty, +\infty)$.

Definition 4. The solution $\eta(t)$ is called **orbitally stable** as $t \rightarrow \infty$ if, for some t_0 , for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every solution $\mathbf{y}(t)$ ($t_0 \leq t < \infty$) satisfying the condition

$$\|\mathbf{y}(t_0) - \eta(t_0)\| < \delta,$$

the inequality

$$\rho(\mathbf{y}(t), L_\eta^+) = \inf_{t_0 \leq t_1 < \infty} \|\mathbf{y}(t) - \eta(t_1)\| < \varepsilon$$

will be ensured for $t_0 \leq t < \infty$, where L_η^+ is the positive semitrajectory of the solution $\eta(t)$.

If, moreover, for $\|\mathbf{y}(t_0) - \eta(t_0)\| < \Delta$, where $\Delta > 0$ is sufficiently small, we have $\rho(\mathbf{y}(t), L_\eta^+) \rightarrow 0$ as $t \rightarrow \infty$, then the solution $\eta(t)$ is called **asymptotically orbitally stable** as $t \rightarrow \infty$.

Definition 5. We shall say that the solution $\eta(t)$ has an **asymptotic phase** ⁽³⁾ if, for every solution $\mathbf{y}(t)$ sufficiently close to it at $t = t_0$, there exists a constant number c (the phase) such that

$$\lim_{t \rightarrow \infty} \|\mathbf{y}(t + c) - \eta(t)\| = 0.$$

Obviously, an orbitally stable solution that has an asymptotic phase is asymptotically orbitally stable.

Main theorem. Suppose the autonomous system (5) admits a solution $\eta(t)$, bounded on I_t , such that

$$\inf_t \|\dot{\eta}(t)\| > 0,$$

and suppose that the variational equation for this solution,

$$dx/dt = \mathbf{f}'(\eta(t))x \quad (6)$$

form a regular linear system, among whose characteristic exponents only one is zero, corresponding to the completely unbounded solution $\mathbf{x}_0 = \vec{\eta}(t)$, while the

others are negative. Then the solution $\vec{\eta}(t)$ is asymptotically orbitally stable as $t \rightarrow \infty$ and has an asymptotic phase.

To prove the main theorem, by means of the change of variables

$$\mathbf{y} = \vec{\eta}(t) + \mathbf{z}$$

we bring system (5) to the form

$$dz/dt = f'(\vec{\eta}(t))z + \vec{\varphi}(t, z), \quad (7)$$

where

$$\begin{aligned} \|\vec{\varphi}(t, \tilde{z}) - \vec{\varphi}(t, z)\| &\leq \\ &\leq N \max(\|z\|, \|\tilde{z}\|) \|\tilde{z} - z\| \end{aligned}$$

for $\|z\|, \|\tilde{z}\| < h < \infty$ (N is a constant). To find a family of solutions $z(t, a)$ of equation (7) such that $z(t, a) \rightarrow 0$ as $t \rightarrow \infty$, where $a = \text{colon}(0, a_2, \dots, a_n)$ is a vector parameter, we use the singular integral equation

$$z(t, a) = X(t)a + \int_{t_0}^{\infty} G(t, \tau) \vec{\varphi}(\tau, z(\tau, a)) d\tau \quad (t \geq t_0), \quad (8)$$

where $X(t)$ is a suitable fundamental matrix of the linear system (6) and

$$G(t, \tau) = \begin{cases} X(t)AX^{-1}(\tau), & \tau > t \geq t_0, \\ X(t)BX^{-1}(\tau), & t_0 \leq \tau < t; \end{cases}$$

$$A = -\text{diag}(E_1, 0), \quad B = \text{diag}(0, E_{n-1})$$

(E_m is the identity matrix of order m). The integral equation (8) is solved by the method of successive approximations analogously to (4).

Moscow State University
named after M. V. Lomonosov

Received
11 VIII 1966

CITED LITERATURE

1. A. Andronov, A. Witt, ZhETF, 3, no. 5 (1933).
2. A. M. Lyapunov, *The General Problem of the Stability of Motion*, L.—M., 1935.
3. O. Vejvoda, Czechoslovak Mathematical Journal, 9 (84), 390 (1959).
4. E. A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations*, IL, 1958.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.