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Abstract

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MATHEMATICS

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ON A MULTIPOINT BOUNDARY-VALUE PROBLEM

(Presented by Academician A. Yu. Ishlinskii, 13 XII 1966)

Consider the equation

$$x^{(n)} + p_1(t)x^{(n-1)} + \dots + p_n(t)x = 0 \quad (a \leq t \leq b), \quad (1)$$

where $p_1(t), \dots, p_n(t)$ are real continuous functions on the interval $[a, b]$,

$$|p_i(t)| \leq L_i, \quad i = 1, 2, \dots, n \quad (a \leq t \leq b),$$

and the multipoint boundary-value problem

$$x(a_i) = A_{i,1}, \quad x'(a_i) = A_{i,2}, \dots, x^{(r_i-1)}(a_i) = A_{i,r_i},$$

$$i = 1, 2, \dots, m \quad (2 \leq m \leq n, \quad r_1 + r_2 + \dots + r_m = n), \quad (2)$$

$$a \leq a_1 < a_2 < \dots < a_m \leq b.$$

A number of works ⁽¹⁻⁵⁾ are devoted to estimating the length of the interval $[a, b]$ on which problem (1)–(2) has a unique solution. The following are known: the Vallée-Poussin estimate ⁽¹⁾

$$\sum_{l=1}^n L_l \frac{(b-a)^l}{l!} \leq 1; \quad (3)$$

the estimate of A. Yu. Levin ⁽²⁾

$$\sum_{k=1}^n \frac{L_k (b-a)^k}{2^k k \left[\frac{k-1}{2} \right]! \left[\frac{k}{2} \right]!} \leq 1, \quad (4)$$

and, finally, Nehari's integral estimate ⁽³⁾

$$\sum_{k=0}^{n-1} \left(\frac{b-a}{2} \right)^k \int_a^b |p_{k+1}(t)| dt \leq 2. \quad (5)$$

Let us note that, for a fixed value of $(b-a)$, conditions (3)–(5) express the requirement of smallness of the coefficients L_i , or respectively

$$\int_a^b |p_i(t)| dt, \quad i = 1, 2, \dots, n.$$

In the present paper each of the estimates (3)–(5) is sharpened. Conditions are obtained under which the coefficient $p_1(t)$ may assume arbitrarily large values on the interval $[a, b]$ owing to the smallness of the remaining coefficients, something not allowed by inequalities (3)–(5).

Theorem 1. *Problem (1)–(2), for arbitrary $A_{i,k}$, has a unique solution on the interval $[a, b]$ if*

$$\sum_{k=2}^n L_k \frac{(b-a)^k}{k!} \exp\{L_1(b-a)\} \leq 1. \quad (6)$$

Estimate (6) sharpens estimate (3). Indeed, let us rewrite (3) and (6), respectively, in the form

$$\sum_{k=2}^n L_k \frac{(b-a)^k}{k!} \leq 1 - L_1(b-a), \quad (7)$$

$$\sum_{k=2}^n L_k \frac{(b-a)^k}{k!} \leq \exp\{-L_1(b-a)\}. \quad (8)$$

The right-hand side of inequality (8), for any value of $(b-a)$, is greater than the right-hand side of inequality (7). Therefore inequality (6) determines a larger interval of solvability of problem (1)–(2) than inequality (3).

A consequence of Theorem 1 is the assertion that, for the equation

$$x^{(n)} + p_1(t)x^{(n-1)} = 0$$

problem (1)–(2) is solvable on any interval $[a, b]$, where the function $p_1(t)$ is continuous.

Theorem 2. *Problem (1)–(2), for arbitrary $A_{i,k}$, has a unique solution on the interval $[a, b]$, if*

$$\sum_{k=2}^n \frac{L_k(b-a)^k}{2^k k \left[\frac{k-1}{2}\right]! \left[\frac{k}{2}\right]!} \exp \left\{ \frac{L_1}{2} (b-a) \right\} \leq 1. \quad (9)$$

Inequality (9) determines a larger interval of solvability of problem (1)–(2) than inequality (4).

Theorem 3. *Problem (1)–(2), for arbitrary A_{ik} , has a unique solution on the interval $[a, b]$, if*

$$\sum_{k=1}^{n-1} \frac{(b-a)^k \int_a^b |p_{k+1}(t)| dt}{2^k k \left[\frac{k-1}{2}\right]! \left[\frac{k}{2}\right]!} \exp \left\{ \frac{1}{2} \int_a^b |p_1(t)| dt \right\} \leq 2. \quad (10)$$

Inequality (10) is a strengthening of inequality (5).

Corollary. *If a nontrivial solution of the equation*

$$x'' + p_1(t)x' + p_2(t)x = 0 \quad (11)$$

has two zeros on the interval $[a, b]$, then

$$(b-a) \int_a^b |p_2(t)| dt \exp \left\{ \frac{1}{2} \int_a^b |p_1(t)| dt \right\} > 4. \quad (12)$$

If in equation (11) $p_1(t) = 0$, then inequality (12) passes into Lyapunov's inequality

$$(b-a) \int_a^b |p_2(t)| dt > 4.$$

Therefore Theorem 3 may be regarded as a generalization of Lyapunov's inequality for an equation of order n .

In conclusion we give the proof of Theorem 3. Suppose the contrary: let inequality (10) hold and, at the same time, let some nontrivial solution $x(t)$ of equation (1) have on the interval $[a, b]$ at least n zeros. Then on the interval $[a, b]$ there will be found a system of points

$$a \leq a_1 \leq a_2 \leq \dots \leq a_{n-1} \leq c \leq b_{n-1} \leq \dots \leq b_1 \leq b,$$

at which

$$x(a_1) = x'(a_2) = \dots = x^{(n-2)}(a_{n-1}) = x^{(n-1)}(c) = \dots = x(b_1) = 0.$$

Let

$$\sup_{a \leq t \leq c} |x^{(n-1)}(t)| = |x^{(n-1)}(\alpha)| = \mu \quad (a \leq \alpha \leq c).$$

Then on the interval $[a, c]$ the estimate (2) holds:

$$|x^{(n-k-1)}(t)| \leq \mu \frac{(c-a)^k}{k \left[\frac{k-1}{2}\right]! \left[\frac{k}{2}\right]!}, \quad k = 1, 2, \dots, n-1. \quad (13)$$

We rewrite equation (1) in the form

$$\left\{ x^{(n-1)}(t) \exp \left[\int_a^t p_1(\xi) d\xi \right] \right\}' = - \sum_{k=1}^{n-1} p_{k+1}(t) x^{(n-k-1)}(t) \exp \left[\int_a^t p_1(\xi) d\xi \right].$$

Integrating this equation from a to c and applying estimate (13), we obtain the inequality

$$1 < \sum_{k=1}^{n-1} \frac{(c-a)^k \int_a^c |p_{k+1}(t)| dt}{k \left[\frac{k-1}{2}\right]! \left[\frac{k}{2}\right]!} \exp \int_a^c |p_1(\xi)| d\xi. \quad (14)$$

Similarly, for the interval $[c, b]$ we have

$$1 < \sum_{k=1}^{n-1} \frac{(b-c)^k \int_c^b |p_{k+1}(t)| dt}{k \left[\frac{k-1}{2}\right]! \left[\frac{k}{2}\right]!} \exp \int_c^b |p_1(\xi)| d\xi. \quad (15)$$

Denote the k -th term of the sums appearing on the right-hand sides of inequalities (14) and (15), respectively, by A_k^{1+k} , B_k^{1+k} , and introduce the quantity

$$\chi = \frac{1}{2} \int_a^b |p_1(t)| dt - \int_a^c |p_1(t)| dt.$$

Then

$$\frac{\int_a^b |p_{k+1}(t)| dt}{k \left[\frac{k-1}{2}\right]! \left[\frac{k}{2}\right]!} \exp \left\{ \frac{1}{2} \int_a^b |p_1(t)| dt \right\} = \frac{[A_k \exp \frac{\chi}{1+k}]^{1+k}}{(c-a)^k} +$$

$$+ \frac{[B_k \exp(-\frac{\chi}{1+k})]^{1+k}}{(b-c)^k} \geq \frac{[A_k \exp \frac{\chi}{1+k} + B_k \exp(-\frac{\chi}{1+k})]^{1+k}}{(b-a)^k}$$

or

$$\begin{aligned} & \sum_{k=1}^{n-1} \frac{(b-a)^k \int_a^b |p_{k+1}(t)| dt}{2^k k \left[\frac{k-1}{2}\right]! \left[\frac{k}{2}\right]!} \exp \left\{ \frac{1}{2} \int_a^b |p_1(t)| dt \right\} \geq \\ & \geq \sum_{k=1}^{n-1} 2^{-k} \left[A_k \exp \frac{\chi}{1+k} + B_k \exp \left(-\frac{\chi}{1+k} \right) \right]^{1+k} > 2 \end{aligned} \quad (16)$$

under conditions (14) and (15), i.e., under the condition that

$$\sum_{k=1}^{n-1} A_k^{1+k} > 1, \quad \sum_{k=1}^{n-1} B_k^{1+k} > 1.$$

Inequality (16) contradicts inequality (10). The theorem is proved.

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