

LIMITS OF UNCERTAINTY IN THE SENSE OF $\backslash(T\backslash)$ -MEANS FOR TRIGONOMETRIC SERIES

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Abstract

Full Text

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MATHEMATICS

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LIMITS OF UNCERTAINTY IN THE SENSE OF T -MEANS FOR TRIGONOMETRIC SERIES

Let us take an infinite matrix

$$T = \|a_{mk}\| \quad (m, k = 0, 1, 2, \dots) \quad (1)$$

and some infinite series

$$\sum_{n=0}^{\infty} u_n \quad (2)$$

with partial sums S_m ($m = 0, 1, 2, \dots$). We shall call the quantities

$$A_m(T) = \sum_{k=0}^{\infty} a_{mk} S_k \quad (m = 0; 1, 2, \dots) \quad (3)$$

the T -means of the series (2), determined by the matrix (1).

It is said that the series (2) is summable by the method T to the value S , if the series on the right-hand side of equality (3) converge for every $m = 0, 1, 2, \dots$ and the quantities $A_m(T)$ tend to the limit S as $m \rightarrow \infty$. The summation method T is called regular if every series (2) converging to a finite value S is summable by the method T to the same value S .

The summation method T is called a method with finite rows if the matrix (1) defining this method satisfies the condition

$$a_{mk} = 0 \quad (k > \nu_m, m = 0, 1, 2, \dots), \quad (4)$$

where ν_m are natural numbers, in general depending on m .

Theorem 1. *Let an arbitrary regular summation method T with finite rows be given, determined by the matrix (1). Then, for any two measurable functions $F(x)$ and $G(x)$ satisfying the inequality*

$$G(x) \leq F(x) \quad (5)$$

almost everywhere on the interval $[-\pi, \pi]$, one can define a trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \quad (6)$$

which has the following properties:

1) If

$$A_m(x, T) \quad (m = 0, 1, 2, \dots) \quad (7)$$

*are the T -means of the series (6), determined by the matrix (1), then the upper limit in measure on $[-\pi, \pi]$ of the sequence (7) is equal to $F(x)$, and the lower limit in measure on $[-\pi, \pi]$ of the same sequence is equal to $G(x)$ **. *

2)

$$\lim_{n \rightarrow \infty} a_n = 0, \quad \lim_{n \rightarrow \infty} b_n = 0. \quad (8)$$

* The functions $F(x)$ and $G(x)$ may be equal to $+\infty$ or $-\infty$ on sets of positive measure.

** The definition of upper and lower limits in measure of a sequence of functions is given in (1), p. 4.

If in Theorem 1 we assume that $F(x) = G(x)$ almost everywhere on $[-\pi, \pi]$, then we obtain the following theorem.

Theorem 2. *For any measurable function $F(x)$, defined almost everywhere on the segment $[-\pi, \pi]$, one can define a trigonometric series (6), satisfying condition (8), which converges in measure on $[-\pi, \pi]$ to the function $F(x)$.*

In the proof of Theorem 1 the following is used.

Lemma A. *Let there be given: an arbitrary regular summability method T with finite rows, defined by the matrix (1); an arbitrary function $\varphi(x)$, continuous on the segment $[-\pi, \pi]$; an arbitrary positive number $\sigma < 1$, and arbitrary natural numbers L and L_0 .*

Then one can define a natural number $L' > L$, a trigonometric polynomial

$$H(x) = \sum_{j=L+1}^{L'} (a_j \cos jx + b_j \sin jx)$$

and sets E, G_n ($L < n \leq L'$), which satisfy the following conditions:

- 1) $\text{mes } E < \sigma, E \subset [-\pi, \pi];$
- 2) $\text{mes } G_n < \sigma, G_n \subset [-\pi, \pi] (L < n \leq L');$
- 3) if $\tau_k(x)$ and $B_n(x, T)$ are determined from the equalities

$$\tau_k(x) = \begin{cases} 0 & (0 \leq k \leq L), \\ \sum_{j=L+1}^k (a_j \cos jx + b_j \sin jx) & (L < k \leq L'), \\ H(x) & (k > L'), \end{cases}$$

$$B_n(x, T) = \sum_{k=0}^{\infty} a_{nk} \tau_k(x) \quad (n = 0, 1, 2, \dots),$$

then

$$|B_n(x, T) - \varphi(x)| < \sigma \quad (x \in [-\pi, \pi] - E, n \geq L');$$

$$4) B_n(x, T) = \theta_n(x) \varphi(x) + \eta_n(x) \quad (x \in [-\pi, \pi] - G_n, L < n \leq L'),$$

where $|\theta_n(x)| < K, |\eta_n(x)| < \sigma$ ($x \in [-\pi, \pi] - G_n, L < n \leq L'$), and K is a constant depending only on the summability method T ;

$$5) B_n(x, T) = 0 \quad (0 \leq n \leq L);$$

$$6) L_0 < L', \sqrt{a_n^2 + b_n^2} < \sigma \quad (L < n \leq L').$$

Definition of a trigonometric series (6) satisfying the conditions of

Theorem 1. Let us take an arbitrary summability method T and arbitrary functions $F(x), G(x)$ satisfying the conditions of Theorem 1, and put

$$f_{2\nu}(x) = F(x), \quad f_{2\nu+1}(x) = G(x) \quad (\nu = 0, 1, 2, \dots).$$

Take a sequence of functions $h_m(x)$ ($m = 0, 1, 2, \dots$), continuous on $[-\pi, \pi]$, which is an almost uniformly convergent sequence to the sequence of functions $f_m(x)$ ($m = 0, 1, 2, \dots$) almost everywhere on $[-\pi, \pi]$. Put, further,

$$u_m(x) = \max[h_m(x), h_{m+1}(x)], \quad v_m(x) = \min[h_m(x), h_{m+1}(x)]$$

$$(m = 0, 1, 2, \dots),$$

$$w_{2\nu}(x) = v_\nu(x), \quad w_{2\nu+1}(x) = u_\nu(x), \quad (\nu = 0, 1, 2, \dots),$$

$$\mu_m = [2^m \Omega_m] = 1 \quad (m = 1, 2, \dots),$$

* If the function $F(x)$ is finite almost everywhere on $[-\pi, \pi]$, then Theorem 2 is easily obtained from previously known theorems.

** The definition of almost uniformly convergent sequences is given in (1), p. 27. The existence of a sequence of functions $h_m(x)$ with the stated properties follows from Lemma 3.3 in (1), p. 27.

where $[a]$ denotes the integer part of the number a , and

$$\Omega_m = \max_{x \in [-\pi, \pi]} |w_m(x) - w_{m-1}(x)| \quad (m = 1, 2, \dots),$$

$$r_0 = 0, \quad r_m = \sum_{s=1}^m \mu_s \quad (m = 1, 2, \dots),$$

$$Q_0(x) = 0, \quad Q_t(x) = w_{m-1}(x) + (t - r_{m-1}) \frac{w_m(x) - w_{m-1}(x)}{\mu_m}$$

$$(r_{m-1} < t \leq r_m, \quad m = 1, 2, \dots).$$

It is easy to see that the functions $Q_t(x)$ are continuous on $[-\pi, \pi]$ and are defined for all $t = 0, 1, 2, \dots$. We now define an increasing sequence of natural numbers M_t ($t = 0, 1, 2, \dots$) and the trigonometric series (6) in the following way.

Put $M_0 = 1$, $a_0 = a_1 = b_1 = 0$. Suppose next that the natural number M_{t-1} and the numbers a_j, b_j for $j = 1, 2, \dots, M_{t-1}$, where t is some natural number, have already been defined. Put

$$S_{t-1,0}(x) = 0, \quad S_{t-1,k}(x) = \begin{cases} \sum_{j=1}^k (a_j \cos jx + b_j \sin jx), & (1 \leq k \leq M_{t-1}), \\ \sum_{j=1}^{M_{t-1}} (a_j \cos jx + b_j \sin jx), & (M_{t-1} < k), \end{cases}$$

$$D_{t-1,n}(x, T) = \sum_{k=0}^{\infty} a_{nk} S_{t-1,k}(x).$$

It is easy to see that, for a given t , the sequence of functions $D_{t-1,n}(x, T)$ ($n = 0, 1, 2, \dots$) converges uniformly on $[-\pi, \pi]$ to the function

$$S_{t-1, M_{t-1}}(x) = \sum_{j=1}^{M_{t-1}} (a_j \cos jx + b_j \sin jx),$$

as $n \rightarrow \infty$. In this case we can define a natural number M'_t satisfying the conditions

$$M'_t > M_{t-1}, \quad \gamma_t < 1/2^t,$$

where

$$\gamma_t = \sup_{n, n' > M'_t} \max_{x \in [-\pi, \pi]} |D_{t-1, n}(x, T) - D_{t-1, n'}(x, T)|.$$

We now apply Lemma A, in which we put

$$\varphi(x) = Q_t(x) - D_{t-1, M_{t-1}}(x, T), \quad (9)$$

$$\sigma = 1/2^t, \quad L = M_{t-1}, \quad L_0 = M'_t. \quad (10)$$

Then, on the basis of this lemma, we can define a natural number $M_t > M_{t-1}$, a trigonometric polynomial

$$H_t(x) = \sum_{j=M_{t-1}+1}^{M_t} (a_j \cos jx + b_j \sin jx) \quad (11)$$

and the sets

$$E_t, \quad G_{t, n} \quad (M_{t-1} < n \leq M_t),$$

which satisfy all the conditions of Lemma A, if they are taken correspondingly-respectively, instead of L' , $H(x)$, E , and G_n , and define $\varphi(x)$, σ , L , and L_0 from equalities (9) and (10).

Thus we define trigonometric polynomials $H_t(x)$ for all $t = 1, 2, \dots$, where the natural numbers M_t ($t = 0, 1, 2, \dots$) form an increasing sequence. Since we have set $M_0 = 1$, $a_0 = a_1 = b_1 = 0$, all terms of the trigonometric series (6) will be determined. It can be shown that the trigonometric series thus obtained satisfies all the conditions of Theorem 1.

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Note: Figure translations are in progress. See original paper for figures.

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