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Abstract

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CYBERNETICS AND CONTROL THEORY

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ON THE PROBLEM OF THE ENCOUNTER OF MOTIONS

Consider the problem (¹⁻⁶) of the minimax time to encounter of the pursuing ($y(t)$) and pursued ($z(t)$) controlled motions, described respectively by the equations

$$\dot{y} = Ay + Bu, \quad \dot{z} = Az + Bv \quad (1)$$

under the condition that the resources of the controls $u(t)$ and $v(t)$, which may be used for $t \geq \tau$ for each current time instant τ , are constrained by the integral conditions

$$\int_{\tau}^{\infty} \|u(t)\|^2 dt \leq \mu^2(\tau), \quad \int_{\tau}^{\infty} \|v(t)\|^2 dt \leq \nu^2(\tau). \quad (2)$$

Here y, z are n -vectors; u, v are r -vectors; A, B are constant matrices of the corresponding dimensions. The quantities $\mu(\tau)$ and $\nu(\tau)$ change with increasing τ in accordance with the actual expenditure of the control resources. This leads to the equations

$$\dot{\mu} = -\|u\|^2/2\mu, \quad \dot{\nu} = -\|v\|^2/2\nu. \quad (3)$$

The control $u(\tau)$ at each time instant τ , according to the conditions of the problem, must be formed by the feedback principle on the basis of measuring the quantities $y(\tau), z(\tau), \mu(\tau)$, and $\nu(\tau)$, i.e.

$$u(\tau) = u[y(\tau), z(\tau), \mu(\tau), \nu(\tau)]. \quad (4)$$

At the same time, information about the future choice of $v(t)$ ($t \geq \tau$) is absent. For $v(\tau)$, both program control and control by the feedback principle are allowed, i.e. in the form

$$v(\tau) = v[y(\tau), z(\tau), \mu(\tau), \nu(\tau)]. \quad (5)$$

If the question is that of encounter in all phase coordinates $\{y_i\}$ and $\{z_i\}$, i.e., if it is required at the encounter instant $t = \tau + T$ to achieve the equality $y(\tau + T) = z(\tau + T)$, then the problem of $\min_u \max_v T$ is solved ⁽⁶⁾ by the rule of extremal aiming ⁽⁵⁻⁷⁾. This rule is as follows. From the measured $y(\tau), z(\tau), \mu(\tau), \nu(\tau)$, the attainability regions $G_1[\tau, T], G_2[\tau, T]$ of the motions $y(t)$ and $z(t)$ are constructed, and the absorption instant $\tau + T^0(\tau)$ is found, when for the first time $G_2 \subset G_1$. Then $\min_u \max_v T = T^0$, and the optimal controls $u^0(\tau), v^0(\tau)$, formed by the feedback principle, are determined from the condition of program aiming of the motions $y(t)$ and $z(t)$ ($t \geq \tau$) to the point ξ^* , where the boundaries of the regions G_1 and G_2 touch. We emphasize that, owing to the integral character of the constraints (2), the corresponding differential game does not, generally speaking, have a saddle point, since the extremal control v^0 does not ensure a time to encounter no less than T^0 (see Example 7.1 in the article ⁽⁶⁾).

The rule of extremal aiming is equivalent ⁽⁶⁾ to the rule: to the original problem there is assigned a problem of limiting speed,

$$y(\tau) - z(\tau) = x(\tau) \rightarrow x(\tau + T^0) = 0, \quad T^0 = \min_w T \quad (6)$$

for the system

$$\dot{x} = Ax + Bw \quad (7)$$

under the condition

$$\xi(\tau) = \left[\int_{\tau}^{\infty} \|w(t)\|^2 dt \right]^{1/2} \leq \mu(\tau) - \nu(\tau); \quad (8)$$

then $T^0 = \min_u \max_v T$, and the optimal controls u^0, v^0 are determined by the equalities

$$u^0(\tau) = \frac{\mu}{\xi} w_{\tau}^0(\tau), \quad v^0(\tau) = \frac{\nu}{\xi} w_{\tau}^0(\tau), \quad (9)$$

where $w_{\tau}^0(t)$ is the solution of problem (6), (7) under condition (8).

In the case where the meeting is considered only with respect to part of the coordinates $\{y_{i_j}\} = y_{[m]}, \{z_{i_j}\} = z_{[m]} (j = 1, \dots, m < n)$, when at the moment of meeting it is required only that $y_{[m]}(\tau + T) = z_{[m]}(\tau + T)$, the situation becomes more complicated. The analysis of this situation constitutes the content of the present note.

The following assertions are valid:

1. The choice of the control $u(\tau)$ from the condition of extremal aiming does not guarantee a meeting of the motions $y(t)$ and $z(t)$ in the time $\tau \leq t \leq \tau + T^0$, and this aiming does not ensure $\min_u \max_v T = T^0$.

(The reachability regions G_1 and G_2 and the point ξ^* are now constructed, naturally, in the subspace $\{x_{i_j}\}$ ($j = 1, \dots, m$).) This assertion is verified, for example, in the case of the motions

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = u_1, \quad \dot{y}_3 = y_4, \quad \dot{y}_4 = u_2, \quad (10)$$

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = v_1, \quad \dot{z}_3 = z_4, \quad \dot{z}_4 = v_2,$$

when it is required to bring about a meeting only in the coordinates y_1, y_3 and z_1, z_3 . If in (10) $u = u^0(\tau)$ is determined by the rule of extremal aiming, and $v(\tau) = 0$, then for the initial data $y_1(0) = -0.25$, $y_2(0) = y_3(0) = y_4(0) = 0$, $\mu(0) = 1 + 2\sqrt{3}$, $z_1(0) = z_2(0) = z_3(0) = z_4(0) = 0$, $\nu(0) = 1$, even before the coincidence of $y_1(t), y_3(t)$ and $z_1(t), z_3(t)$, at the moment $\tau_* = 0.144$ the regions G_1 and G_2 merge, and at this moment the motion $z(t)$ obtains the possibility of slipping out of the reachability region $G_1(\tau, T)$ of the pursuing motion $y(t)$.

2. It is impossible in general to construct a control $u(\tau) = u^*[y(\tau), z(\tau), \mu(\tau), \nu(\tau)]$ that would guarantee a meeting at a moment $t \leq \tau + T^0(\tau)$. The assertion is verified by choosing the control $v^* = \frac{\nu}{\mu} u^*$.
3. Let $\varepsilon(\tau)$ and $T_\varepsilon^0(\tau)$ denote the quantities connected by the condition that $T_\varepsilon^0(\tau)$ is the solution of the problem

$$y(\tau) - z(\tau) = x(\tau) \rightarrow x_{[m]}(\tau + T_\varepsilon^0) = 0, \quad T_\varepsilon^0 = \min_w T_\varepsilon \quad (11)$$

for system (7) under the condition

$$\xi(\tau) = \left[\int_\tau^\infty \|w(t)\|^2 dt \right]^{1/2} \leq \mu(\tau) - \nu(\tau) - \varepsilon(\tau). \quad (12)$$

Then one can specify a control $u = u_0[y(\tau), z(\tau), \mu(\tau), \nu(\tau), \varepsilon(\tau)]$, ensuring a meeting no later than at the moment $t = \tau_0 + T_\varepsilon^0(\tau_0)$, whatever the initial data $y(\tau_0), z(\tau_0), \mu(\tau_0), \nu(\tau_0), \varepsilon(\tau_0) > 0$, for which problem (11), (7), (12) has a solution.

This assertion is verified by constructing the control u_0 in the form

$$u(\tau) = u_0[y(\tau), z(\tau), \mu(\tau), \nu(\tau), \varepsilon(\tau)] = w_\tau^0(\tau) \varphi[\zeta, \varepsilon]. \quad (13)$$

Here $w_\tau^0(t)$ is the solution of problem (11), (7) under condition (12). The number $T_\varepsilon^0(\tau)$ is computed in the course of the process as the least of the numbers T_ε^0 for which $\varepsilon(\tau) = \min[\varepsilon(\tau - 0), \varepsilon_0]$, where $\varepsilon_0 > 0$ is a number chosen in advance.

Finally, the function $\varphi[\zeta, \varepsilon]$ is chosen so as to ensure $\varepsilon(\tau) > 0$ for $t < \tau + T_\varepsilon^0(\tau)$. In particular, the functions

$$\varphi[\zeta, \varepsilon] = \left\{ \frac{\nu + \zeta}{\varepsilon\sqrt{\zeta}} \quad \text{for } 0 \leq \zeta < \varepsilon^2, \quad \frac{\nu + \zeta}{\zeta} \quad \text{for } \zeta \geq \varepsilon^2 \right\},$$

$$\varphi[\zeta, \varepsilon] = \left\{ \frac{\nu + \zeta}{\beta\varepsilon} \quad \text{for } 0 \leq \zeta < \beta\varepsilon, \quad \frac{\nu + \zeta}{\zeta} \quad \text{for } \zeta \geq \beta\varepsilon \right\},$$

$$\beta = \text{const} > 0.$$

Remark. The meaning of the control u_0 (13) is as follows: by means of a small additional reserve $\varepsilon(\tau) > 0$, a spring damping layer is, as it were, introduced between the boundaries of the regions G_1 and G_2 , not allowing the region G_2 to merge with the region G_1 and, consequently, preventing the region G_2 from going beyond G_1 (see, in this connection, § 5 in paper (8)). The change of $T_\varepsilon^0(\tau)$ in connection with the change of $\zeta(\tau)$ is relatively irregular in character (although the quantity $T_\varepsilon^0(\tau)$ does not increase monotonically). Therefore the control (13), generally speaking, does not close the system by differential feedback (in the sense in which this is defined in paper (6)), and the above-indicated method of computing the quantity $T(\tau)$ should be strictly understood as the limiting case of changing the values of $\tau + T_\varepsilon^0(\tau)$ at discrete instants of time τ_k ($k = 0, 1, \dots$) on small intervals $[\tau_k, \tau_{k+1})$, between which the control (13) is determined with $\tau + T_\varepsilon^0(\tau) = \tau_k + T_\varepsilon^0(\tau_k)$ constant, while the quantity $\zeta(\tau)$ is then determined as the solution of the problem $y(\tau) - z(\tau) = x(\tau) \rightarrow x_{[m]}(\tau_k + T_\varepsilon^0(\tau_k)) = 0$ for system (7) under the condition

$$\zeta(\tau) = \left[\int_{\tau}^{\tau_k + T_\varepsilon^0(\tau_k)} \|w_\tau^0(t)\|^2 dt \right]^{1/2} = \min.$$

With the exception of the indicated circumstance, the definition of the class of admissible motions $y(t)$ and $z(t)$ does not essentially go beyond the apparatus of ordinary differential equations. Let us also note that only for exceptional, sufficiently unreasonable (from the standpoint of the interests of the motion $z(t)$) choices of the control $v(\tau)$ with $u = u_0(\tau)$ (13) is it possible for a kind of sliding regime to appear at $\zeta = 0$, which, however, leads to a rapid decrease of the quantity $T_\varepsilon^0(\tau)$.

Assertions (2) and (3) show that the control u_0 (13) ensures, in a certain sense, the maximum of what can be achieved from the standpoint of the motion $y(t)$ by means of an arbitrarily small increase in the control resource $u(\tau)$, regularizing

the problem while preserving the condition of coincidence of the coordinates $\{y_{i_j}\}$ and $\{z_{i_j}\}$ at the moment of encounter. An analogous regularization is also achieved by bringing the point $\{y_{i_j}(t)\}$ at the moment of encounter into a small neighborhood of the point $\{z_{i_j}(t)\}$, but now without changing the initial reserve μ .

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Note: Figure translations are in progress. See original paper for figures.

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