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1967

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Abstract

Full Text

UDC 519.46

MATHEMATICS

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ONE CLASS OF SYMMETRIC SPACES WITH AN EXTENDABLE GROUP OF MOTIONS AND A GENERALIZATION OF THE POINCARÉ MODEL

(Presented by Academician I. G. Petrovskii, May 17, 1966)

Let M be a symmetric space of affine connection, $\mathcal{G}(M)$ the group generated by displacements along geodesics. Generally speaking, the group $\mathcal{G}(M)$ is maximal in the sense that it is not contained in a broader (in dimension) Lie group of transformations of the space M , distinct from the group of motions. Therefore those cases in which such an extension is possible are of special interest. Below a construction will be indicated for one class of symmetric spaces with extendable group $\mathcal{G}(M)$.

1°. Let us recall some facts about transitive-differential groups of transformations ⁽¹⁾. Let G be the Lie algebra of transformations of a (local) manifold M . Each operator from G is represented in the form of the series

$$a + A_1(x) + A_2(x, x) + \dots, \quad (1)$$

where a is a vector of the tangent space T_0 to the manifold M at the initial point $(0, 0, \dots, 0)$, and $A_k(x, \dots, x)$ is a k -linear symmetric operator acting from $T_0 \times \dots \times T_0$ into T_0 . G is called the **Lie algebra of a transitive-differential group** of order ν (or simply the Lie algebra of a family D^ν) if:

1. together with the series (1), the algebra G also contains all the summands A_k .
2. $a \in G$, where a is an arbitrary vector from T_0 .
3. G contains no operators of order higher than ν (i.e. $A_k = 0$ for $k > \nu$), but contains at least one operator $A_\nu \neq 0$.

We shall be interested only in the case $\nu = 2$. In this case every operator $A(x, x) \in G$ determines in the space T_0 a Jordan algebra J with multiplication law $xy = A(x, y)$, and, along with the operator $A(x, x)$, G contains all operators of the form

$$A(a; x, x) = 2(ax)x - ax^2 \quad (a \in T_0).$$

Conversely, if J is an arbitrary Jordan algebra with multiplication law $A(x, y)$, then there exists at least one Lie algebra G of the family D^2 containing $A(x, x)$ as a quadratic operator; the manifold M on which the corresponding (local) group $\mathcal{G}$ acts may be considered to coincide with J . The minimal Lie algebra of this kind is the linear span of all possible vectors $a \in J$ (operators of order zero), the operators $A(a, x)$ and their commutators (linear operators), and, finally, the quadratic operators $A(x, x)$ and $A(a; x, x)$; we shall say that this Lie algebra is generated by the Jordan algebra J .

If the algebra J is semisimple, then the Lie algebra containing it of the family D^2 is unique.

2°. Let J be an arbitrary Jordan algebra (complex or real) with multiplication law $A(x, y)$, and let S be its linear transformation. Consider the Lie algebra G of the family D^2 generated by J . Operators of the form

$$a + A(S(a); x, x) \quad (a \in J)$$

form a linear subspace $E_S \subset G$. Consider the subalgebra $H_S \subset G$, formed by all linear operators $A(x)$ for which $[A(x), E_S] \subset E_S$. Obviously, $H_S = H_{-S}$.

Theorem. *If S is an involutive automorphism of the Jordan algebra J , then*

$$[E_\sigma, E_\sigma] \subset H_\sigma \quad (\sigma = S \text{ or } -S),$$

and thereby the pair $(H_\sigma + E_\sigma, H_\sigma)$ determines on J the structure of a (local) symmetric space M_σ of an affine connection. If the algebra J contains an identity, then $G_\sigma = H_\sigma + E_\sigma$ is the Lie algebra of the group $\mathcal{G}(M_\sigma)$, generated by shifts along geodesics in the space M_σ .

It is obvious that G_σ is a proper subalgebra of the algebra G ; consequently, the construction indicated above leads to (local) symmetric spaces M with extendable group $\mathcal{G}(M)$. Further, since the identity transformation of the algebra J is its involutive automorphism, at least one such space is associated with every Jordan algebra.

Let us note that the affine connection in M_σ is expressed at an arbitrary point x by the formula

$$\Gamma(x; u, v) = A_\sigma(f(x); u, v),$$

where u, v are two arbitrary vectors at the point x . Here

$$A_\sigma(a; u, v) = A(\sigma(a); u, v), \quad f(x) = -2(x - x^3 + x^5 - \dots),$$

where x^{2k+1} denotes

$$A_\sigma(\dots A_\sigma(A_\sigma(x; x, x); x, x) \dots; x, x),$$

i.e. the k -fold iteration of A_σ . Thus, at each point the connection is given by some Jordan algebra.

3°. With the theorem given above there is associated the following construction, which it is natural to regard as a generalization of the Poincaré model (conformal model) of Lobachevsky space to certain symmetric spaces of nonconstant curvature.

Let M be a homogeneous space with transformation group $\mathcal{G}$ and stationary subgroup $\mathcal{G}'$ (for the point $x_0 \in M$). Suppose that the Lie algebra G of the group $\mathcal{G}$ belongs to the family D^2 . Starting from some quadratic operator $A(x, x)$ of the algebra G , and also from an involutive automorphism S (of the corresponding Jordan algebra), we construct, as above, the subspace E_S and the subalgebra $G_S = H_S + E_S$; denote by $\mathcal{G}_S$ the connected subgroup corresponding to this subalgebra. The orbit of the point x_0 with respect to $\mathcal{G}_S$ is some domain in M ; denote it by M_S . Obviously,

$$M_S = \mathcal{G}_S / \mathcal{G}' \cap \mathcal{G}'$$

is a locally symmetric space of an affine connection.

Analogously, the domain M_{-S} is a locally symmetric space of an affine connection, whose structure is dual to the structure of M_S .

We thus obtain a pair of mutually dual locally symmetric spaces M_S and M_{-S} , realized as domains in

* Square brackets denote the commutator in the Lie algebra.

the ambient space M . The motions of each of them extend to transformations belonging to some group $\mathcal{G}$, acting on M . The stationary subgroups of the initial point in M_S and M_{-S} have one and the same connected component of the identity (the latter corresponds to the subalgebra $H_S = H_{-S}$).

Especially interesting is the case when one of the groups $\mathcal{G}_S$ or $\mathcal{G}_{-S}$ is compact. Let, for example, $\mathcal{G}_S$ be compact. Then the domain M_S coincides with the whole space M and is a compact locally symmetric Riemannian space. The dual space M_{-S} is realized as a domain in M_S , and the motions of this domain are generated by transformations of the whole space M_S belonging to some extension of the group $\mathcal{G}(M_S)$.

In the case when the Jordan algebra J has an identity, the transformations from $\mathcal{G}$ have the invariant

$$(x_1, x_2, x_3, x_4) = \frac{\|x_1 - x_3\|}{\|x_2 - x_3\|} : \frac{\|x_1 - x_4\|}{\|x_2 - x_4\|},$$

where $\|x\|$ is the norm of the element $x \in J$. Therefore the distance $\rho_{-S}(x, y)$ in the space M_{-S} , up to a constant factor, is $|\ln(x, y, a, b)|$, where a, b are the infinitely distant points of the geodesic xy joining x and y in M_S (lying on the boundary of the domain M_{-S}).

The Poincaré model of n -dimensional Lobachevskii space may be regarded as a special case of this construction. As J one should take the n -dimensional simple Jordan algebra with multiplication law

$$xy = (e, x)y + (e, y)x - (x, y)e,$$

where x, y is a positive definite scalar product in the vector space J , and e is a fixed vector of length 1. Here $\|x\| = (x, x)$. The corresponding Lie algebra of the family D^2 is the Lie algebra of the conformal group acting on the n -dimensional sphere M (the space J , completed by the point ∞). Taking as S the operator of reflection in the plane $(e, x) = 0$, we obtain the Lie algebra of the rotation group of the sphere M (in stereographic coordinates $x^1, x^2, \dots, x^n$). The operator $-S$ leads to Lobachevskii space, realized as a hemisphere of the sphere in M .

As a second example let us consider the realization of the noncompact symmetric space $M_{-1} = SO(n, C)/SO(n)$ inside the dual (compact) space $M_1 = SO(n) \times SO(n)/SO(n)$ ($M_1 = SO(n)$). In this case J is the simple Jordan algebra of real skew-symmetric matrices of order n with multiplication law $xy = \frac{1}{2}(xky + ykx)$, where k is a fixed nondegenerate skew-symmetric matrix. Here $\|x\| = \det x$. The corresponding Lie algebra G of the family D^2 is the Lie algebra of matrices of the form

$$t = \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix}$$

(a arbitrary, b, c skew-symmetric matrices of order n), acting in J as follows:

$$t(x) = b + ax + xa^T - xcx.$$

The group $\mathcal{G}$ of transformations corresponding to this Lie algebra is isomorphic to $SO(n, n)$. It consists of matrices of the form

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix},$$

preserving the quadratic form $u_1 u_{n+1} + \dots + u_n u_{2n}$. The homogeneous space M is the set of pairs of matrices (y, z) , defined up to

up to simultaneous multiplication on the right by any nonsingular matrix and satisfying the condition $y^{Tz} + z^{Ty} = 0$. $\mathcal{G}$ acts in M as follows:

$$y' = py + qz, \quad z' = ry + sz.$$

(If $\det z \neq 0$, then $(y, z) \sim (x, e)$, where e is the identity matrix and x is skew-symmetric. The algebra G was specified above by its action on the matrices x .)

The identity operator S corresponds to the space $M_1 = M$, and the operator $-S$ to the space M_{-1} , realized in M_1 as the domain

$$\{(x, e) \mid x^T = -x, \quad xx^T < e\}.$$

The boundary of this domain is determined by the condition $\det(xx^T - e) = 0$.

Received

4 V 1966

CITED LITERATURE

1. I. L. Kantor, DAN, 158, No. 6 (1964).

Note: Figure translations are in progress. See original paper for figures.

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