

ON THE THEORY OF THE METHOD OF FINITE DIFFERENCES FOR SOLVING NONLINEAR BOUNDARY-VALUE PROBLEMS

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Abstract

Full Text

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MATHEMATICS

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ON THE THEORY OF THE METHOD OF FINITE DIFFERENCES FOR SOLVING NON- LINEAR BOUNDARY-VALUE PROBLEMS

(Presented by Academician L. V. Kantorovich on 2 IV 1966)

I. Consider the following Dirichlet problem. Find a function $u(x, y)$, twice continuously differentiable in the closed domain $\overline{D}$, satisfying the differential equation

$$F(x, y, u, u_x, u_y, u_{xx}, u_{yy}) = 0 \quad (1)$$

and assuming continuous values on the piecewise smooth boundary

$$u|_{\Gamma} = \varphi(x, y). \quad (2)$$

Here the function $F(x, y, u, p, q, r, t)$ is continuously differentiable with respect to all its arguments and satisfies the inequalities

$$F_r(x, y, u, p, q, r, t) \geq 0, \quad F_t(x, y, u, p, q, r, t) \geq 0,$$

$$F_r(x, y, u, p, q, r, t) + F_t(x, y, u, p, q, r, t) \geq m > 0,$$

$$F_u(x, y, u, p, q, r, t) \leq 0$$

for $x, y \in \overline{D}$, $-\infty < u, p, q, r, t < +\infty$.

Under these conditions, for the solution of the Dirichlet problem (1), (2), a uniqueness theorem is valid; the existence of the solution is assumed by us.

In the usual way we introduce a square mesh with step h , the mesh domain D_h , and its boundary in the sense of the five-point star Γ_h . We divide the boundary points into two groups $\Gamma_h^{(1)}$ and $\Gamma_h^{(2)}$, $\Gamma_h = \Gamma_h^{(1)} + \Gamma_h^{(2)}$. To $\Gamma_h^{(1)}$ we assign all points of Γ_h that belong to Γ , and to $\Gamma_h^{(2)}$ all the remaining points of Γ_h .

The boundary conditions are approximated in two ways:

A. **Simple transfer.** We set

$$\begin{aligned} u_{ij} &= \varphi(x_i, y_j), \quad \text{if } x_i, y_j \in \Gamma_h^{(1)}, \\ u_{ij} &= \varphi(\bar{x}_i, \bar{y}_j), \quad \text{if } x_i, y_j \in \Gamma_h^{(2)}. \end{aligned} \quad (3)$$

Here $\bar{x}_i, \bar{y}_j$ is the point of Γ nearest to x_i, y_j .

B. **Collatz interpolation.** We set

$$\begin{aligned} u_{i,j} &= \varphi(x_i, y_j), \quad \text{if } x_i, y_j \in \Gamma_h^1, \\ u_{i,j} &= \frac{\delta}{\delta + h} u_{i,j-1} + \frac{h}{\delta + h} \varphi(\bar{x}_i, \bar{y}_j), \quad \text{if } x_j y_j \in \Gamma_h^{(2)}. \end{aligned} \quad (4)$$

(For more details, see, for example, ⁽¹⁾.)

1. As the difference equation approximating the differential equation (1), consider the equation

$$\begin{aligned} \Gamma \left(x_i, y_j, u_{i,j}, \frac{u_{i+1,j} - u_{i-1,j}}{2h}, \frac{u_{i,j+1} - u_{i,j-1}}{2h}, \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}, \right. \\ \left. \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} \right) = 0 \end{aligned} \quad (5)$$

for $x_i y_j \in D_h$.

Theorem 1. Let

$$|F_p(x, y, u, p, q, r, t)| \leq M F_r(x, y, u, p, q, r, t),$$

$$|F_q(x, y, u, p, q, r, t)| \leq M F_t(x, y, u, p, q, r, t)$$

for $x, y \in \bar{D}$, $-\infty < u, p, q, r, t < +\infty$.

Then, for $h < 1/M$, the systems of difference equations (5), (3) and (5), (4) have solutions. These solutions are unique and

$$\lim_{h \rightarrow 0} \max_{x_i, y_j \in \bar{D}_h} |u(x_i, y_j) - u_{ij}| = 0.$$

If, in addition, $u \in C^{(3)}(\overline{D})$, then

$$\max_{x_i, y_j \in \overline{D}_h} |u(x_i, y_j) - u_{ij}| \leq Ch.$$

If $u \in C^{(4)}(\overline{D})$ and Kolmat interpolation is used, then

$$\max_{x_i, y_j \in \overline{D}_h} |u(x_i, y_j) - u_{ij}| \leq C_1 h^2.$$

2. We now consider other ways of approximating the first derivatives u_x and u_y . Namely, if $F_p(\dots) \geq 0$, we replace u_x by $(u_{i,j} - u_{i-1,j})/h$; if $F_p(\dots) \leq 0$, we replace u_x by $(u_{i+1,j} - u_{i,j})/h$. The treatment is analogous for u_y . In all, four cases occur. We state the result for one of them.

Consider the following difference equation approximating the differential equation (1):

$$F\left(x_i, y_j, u_{ij}, \frac{u_{i,j} - u_{i-1,j}}{h}, \frac{u_{i,j} - u_{i,j-1}}{h}, \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}, \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}\right) = 0. \quad (6)$$

Theorem 2. Let

$$F_p(x, y, u, p, q, r, t) \geq 0, \quad F_q(x, y, u, p, q, r, t) \geq 0,$$

$$F_u(x, y, u, p, q, r, t) \leq q_1 < 0$$

for $x, y \in \overline{D}$, $-\infty < u, p, q, r, t < +\infty$.

Then the systems of difference equations (6), (4) and (6), (3) have solutions. These solutions are unique and

$$\lim_{h \rightarrow 0} \max_{x_i, y_j \in \overline{D}_h} |u(x_i, y_j) - u_{ij}| = 0.$$

If, moreover, $u \in C^{(3)}(\overline{D})$, then

$$\max_{x_i, y_j \in \overline{D}_h} |u(x_i, y_j) - u_{ij}| \leq C_3 h.$$

- II. We now consider a more general boundary-value problem. Find a function $u \in C^{(2)}(\overline{D})$ satisfying equation (1) and the boundary condition

$$f(x, y, u, u_x, u_y)|_{\Gamma} = 0. \quad (7)$$

Here the function $f(x, y, u, p, q)$ is continuously differentiable with respect to all its arguments and satisfies the inequalities

$$-f_u(x, y, u, p, q) \geq c > 0,$$

$$f_p(x, y, u, p, q) \cos(\mathbf{n}, x) \geq 0, \quad f_q(x, y, u, p, q) \cos(\mathbf{n}, y) \geq 0$$

for $x, y \in \Gamma$, $-\infty < u, p, q < \infty$; $\mathbf{n}$ is the inward normal to the boundary Γ . We assume that the solution of problem (1), (7) exists and is unique.

We approximate the boundary conditions (7) as follows. Let the point $x_i, y_j \in \Gamma_h$. Draw the inward normal to the boundary Γ passing through the point x_i, y_j . We assume that the domain D is such that, for $h < h_0$

(h_0 does not depend on x_i, y_j) there exists a grid square with vertex at x_i, y_j such that the two vertices of this square adjacent to the vertex x_i, y_j belong to $\bar{D}_h$, and the normal constructed above passes inside or along the boundary of this square. For definiteness, suppose that the vertices indicated above have coordinates x_{i-1}, y_j and x_i, y_{j-1} . Denote by $\bar{x}_i, \bar{y}_j$ the point of intersection of the normal drawn at x_i, y_j with the boundary Γ . Then we set

$$f\left(\bar{x}_i, \bar{y}_j, u_{i,j}, \frac{u_{i,j} - u_{i-1,j}}{h}, \frac{u_{i,j} - u_{i,j-1}}{h}\right) = 0 \quad (8)$$

for $x_i, y_j \in \Gamma_h$.

Theorem 3. Suppose

$$\begin{aligned} F_u(x, y, u, p, q, r, t) &\leq q_1 < 0, \\ |F_p(x, y, u, p, q, r, t)| &\leq MF_r(x, y, u, p, q, r, t), \\ |F_q(x, y, u, p, q, r, t)| &\leq MF_t(x, y, u, p, q, r, t) \end{aligned}$$

for $x, y \in \bar{D}$, $-\infty < u, p, q, r, t < +\infty$.

Then, for sufficiently small h , the system of difference equations (5), (8) has a solution. This solution is unique and

$$\lim_{h \rightarrow 0} \max_{x_i, y_j \in \bar{D}_h} |u(x_i, y_j) - u_{i,j}| = 0.$$

If, moreover, $u \in C^{(3)}(\bar{D})$, then

$$\max_{x_i, y_j \in \bar{D}_h} |u(x_i, y_j) - u_{i,j}| \leq C_4 h.$$

If $u \in C^{(4)}(\bar{D})$, $f_p = f_q = 0$, $\Gamma_h = \Gamma_h^{(1)}$, then

$$\max_{x_i, y_j \in \bar{D}_h} |u(x_i, y_j) - u_{i,j}| \leq C_5 h^2.$$

In the case of sign-definiteness of the derivatives of the function F with respect to the variables u_x and u_y , and approximation of the differential equation in the form (6), theorems analogous to Theorem 2 hold. We state one of them.

Theorem 4. Under the assumptions of Theorem 2, the system of difference equations (6), (8) is uniquely solvable and

$$\lim_{h \rightarrow 0} \max_{x_i, y_j \in \bar{D}_h} |u(x_i, y_j) - u_{i,j}| = 0.$$

If, moreover, $u \in C^{(3)}(\bar{D})$, then

$$\max_{x_i, y_j \in \bar{D}_h} |u(x_i, y_j) - u_{i,j}| \leq C_6 h.$$

Remark. All results carry over to the case of any number of dimensions.

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1. L. Collatz, *Numerical Methods for Solving Differential Equations*, Moscow, 1953.

Note: Figure translations are in progress. See original paper for figures.

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