

ON THE STRUCTURE OF TENSORS ADMITTING A GIVEN GROUP OF INVARIANCE

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Abstract

Full Text

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MATHEMATICS

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ON THE STRUCTURE OF TENSORS ADMITTING A GIVEN GROUP OF INVARIANCE

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Tensors in the complex vector space E^n , generally speaking relative tensors, are considered. The latter means that under a change of basis $\{e_i\}$

$$e_{i'} = A_{i'}^i e_i \quad (1)$$

the law of transformation of the coordinates of the tensor is complicated by multiplication by $(\text{Det } \|A_{i'}^i\|)^p$, where p is an integer (possibly negative), called the weight of the tensor. If $p = 0$, then the tensor is called absolute. A special role is played by the n -vectors $\omega_{i_1 \dots i_n}$, $\varepsilon_{i_1 \dots i_n}$ (skew-symmetric in all indices), where ω has weight -1 and has the same coordinates in every basis, $\omega_{12 \dots n} = 1$; ε is an absolute tensor and is determined up to a constant factor. Analogously one defines $\omega^{i_1 \dots i_n}$, $\varepsilon^{i_1 \dots i_n}$ (in abbreviated notation $\bar{\omega}, \bar{\varepsilon}$).

The **group of invariance** of a given tensor (or finite system of tensors) Z is the subgroup $\mathfrak{G} \subset GL(n, C)$ of all nonsingular matrices $\|A_{i'}^i\|$ for which the transformations (1) of some initial basis $\{e_i^0\}$ do not change the corresponding numerical values of the coordinates of Z (the dependence of $\mathfrak{G}$ on the choice of the initial basis is trivial and is not reflected in the sense of the theory).

A concomitant $K(Z)$ of a system of tensors Z is a tensor obtained from the tensors Z by the operations of addition, multiplication and contraction of tensors, multiplication of a tensor by a number, and permutation of the indices of a tensor (if all indices of the tensors Z are lower (upper), then the operation of contraction, of course, drops out).

Lemma 1. *For every nonzero contravariant absolute tensor Z there exists a nonzero concomitant $K(Z)$ which, in a suitable basis $\{e_i\}$, has the form*

$$K(Z) = e_1 \dots e_1 [e_1 e_2] \dots [e_1 e_2] [e_1 e_2 e_3] \dots [e_1 e_2 e_3] [e_1 e_2 \dots e_n] \dots [e_1 e_2 \dots e_n]. \quad (2)$$

Here e_1 is a vector, i.e. a unicontravariant tensor; $[e_1 e_2]$ is a simple bivector, i.e. a twice contravariant tensor—the alternating product of the tensors e_1, e_2 ;

$[e_1 e_2 e_3]$ is a simple trivector, obtained analogously from e_1, e_2, e_3 , etc. These tensors, each taken some (possibly zero) number of times, are then multiplied among themselves.

Remark. If a nonzero contravariant absolute tensor Z has an irreducible group of invariance $\mathfrak{G}$, then its concomitant (2) has, obviously, the simplified form

$$K(Z) = [e_1 e_2 \dots e_n] \dots [e_1 e_2 \dots e_n]. \quad (3)$$

We note that

$$[e_1 e_2 \dots e_n]^{i_1 \dots i_n} = \varepsilon^{i_1 \dots i_n}.$$

Lemma 1, with an inessential complication, is also true for relative contravariant tensors Z .

All the results hold, of course, also for covariant tensors.

Lemma 2. *Let a (relative) tensor W admit a group of invariance*

of invariance $\mathfrak{G}$ of a system of (relative) tensors Z . Then there exist such concomitants S and $T \neq 0$ of $Z, \omega, \bar{\omega}$, that

$$W \cdot T(Z, \omega, \bar{\omega}) = S(Z, \omega, \bar{\omega}) \quad (4)$$

(i.e., W is, as it were, the quotient of two concomitants).

In particular, if Z and W are covariant tensors, then

$$W \cdot T(Z, \omega) = S(Z, \omega); \quad (5)$$

$\bar{\omega}$ does not enter into the concomitants (and there is no contraction).

If Z and W are absolute tensors, then (4) takes the form:

$$W \cdot T(Z, \delta) = S(Z, \delta), \quad (4')$$

where T and S are concomitants of Z and of the invariant tensor

$$\delta_j^i = \begin{cases} 0 & (i \neq j), \\ 1 & (i = j). \end{cases}$$

In particular, in the case of covariant Z, W ,

$$W \cdot T(Z) = S(Z). \quad (5')$$

Of course, for the case of contravariant Z, W analogous results hold.

Combining Lemma 1 (the remark) and Lemma 2, we obtain the following theorem.

Theorem. If the group of invariance $\mathfrak{G}$ of a system of (relative) tensors Z is irreducible, then every (relative) tensor W admitting the group $\mathfrak{G}$ has the form

$$W = J^p \cdot S(Z, \omega, \bar{\omega}), \quad (6)$$

where J is a relative scalar of weight -1 (one may take $J = \varepsilon^{12\dots n}$); S is a concomitant; p is an integer; J^p necessarily admits the group $\mathfrak{G}$.

In particular, if Z, W are absolute tensors:

$$W = S(Z, \delta) \varepsilon \varepsilon \dots \varepsilon \quad \text{or} \quad = S(Z, \delta) \bar{\varepsilon} \bar{\varepsilon} \dots \bar{\varepsilon}, \quad (7)$$

where the product $\varepsilon \varepsilon \dots \varepsilon$ (or $\bar{\varepsilon} \bar{\varepsilon} \dots \bar{\varepsilon}$) is contracted in all its indices with part of the indices of $S(Z, \delta)$. This product may also be absent (the number of factors ε is equal to 0). If it is present, it must admit the group $\mathfrak{G}$.

Finally, if the absolute tensors Z, W are covariant, then (7) simplifies:

$$W = S(Z) \bar{\varepsilon} \dots \bar{\varepsilon}, \quad (8)$$

and analogously for contravariant Z, W :

$$W = S(Z) \varepsilon \dots \varepsilon. \quad (9)$$

Thus, under the condition of irreducibility of $\mathfrak{G}$, the coordinates of W are expressed through the coordinates of Z by polynomials, and not only by rational functions, as in the general case (Lemma 2).

Example. All covariant absolute tensors in a complex simple Lie algebra, invariant under all its automorphisms, are concomitants of its covariant metric tensor $g_{ij}(= C_{jp}^i C_{iq}^p)$, where C_{jk}^i is the structure tensor) and of the covariant structure tensor $C_{ijk}(= g_{ip} C_{jk}^p)$, possibly also contracted with $\bar{\varepsilon} \dots \bar{\varepsilon}$ over all indices of this product. The number of factors $\bar{\varepsilon}$ is arbitrary if all automorphisms have $\text{Det} = 1$, and even if $\text{Det} = \pm 1$.

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Note: Figure translations are in progress. See original paper for figures.

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