

# ON SOME CRITERIA FOR NONOSCILLATION OF LINEAR DIFFERENTIAL OPERATORS

G. S. Zaitseva

1967

SovietRxiv

---

View the original and related papers at <https://sovietsrxiv.org/items/ru-196701.53265>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

UDC 517.941

**MATHEMATICS**

**G. S. ZAITSEVA**

# ON SOME CRITERIA FOR NONOSCILLATION OF LINEAR DIFFERENTIAL OPERATORS

(Presented by Academician A. Yu. Ishlinskii, May 15, 1967)

The interval  $[a, b]$  is an interval of nonoscillation of the operator

$$L(x) \equiv x^{(n)} + p_1(t)x^{(n-1)} + \dots + p_n(t)x,$$

if every nontrivial solution of the equation  $L[x] = 0$  has on the interval  $[a, b]$  no more than  $(n - 1)$  zeros (zeros are counted according to their multiplicities).

1. **Theorem 1.** Let  $|p_i(t)| \leq L_i$ ,  $i = 1, 2, \dots, n$  ( $a \leq t \leq b$ ). If the inequality

$$\sum_{k=2}^n \frac{L_k}{\left[\frac{k-1}{2}\right]! \left[\frac{k}{2}\right]! L_1^k} \int_0^{L_1(b-a)/2} \tau^{k-1} e^\tau d\tau \leq 1, \quad (1)$$

is satisfied, then the interval  $[a, b]$  is an interval of nonoscillation of the operator

$$L(x) = x^{(n)} + p_1(t)x^{(n-1)} + \dots + p_n(t)x.$$

Inequality (1) determines a larger interval of nonoscillation than inequality (5)

$$\sum_{k=2}^n \frac{L_k(b-a)^k}{k \left[\frac{k-1}{2}\right]! \left[\frac{k}{2}\right]! 2^k} \exp \left\{ L_1 \frac{b-a}{2} \right\} \leq 1, \quad (2)$$

and therefore refines the estimates  $(1-2, 5)$ .

In the proof of Theorem 1 the following is used.

**Theorem 2.** Let an  $n$ -times differentiable function satisfy the conditions

$$x(a_1) = x'(a_2) = \dots = x^{(n-1)}(a_n) = 0 \quad (a \leq a_1 \leq a_2 \leq \dots \leq a_n \leq b),$$

$$|x^{(n)} + p_1(t)x^{(n-1)}| \leq \mu, \quad |p_1(t)| \leq L_1 \quad (a \leq t \leq b).$$

Then the inequality holds

$$|x(t)| < \mu \int_0^{L_1(b-a)} \tau^{n-1} e^\tau d\tau / L_1^n \left[ \frac{n-1}{2} \right]! \left[ \frac{n}{2} \right]!.$$

**Lemma.** If  $x(t)$  on  $[a, b]$  is a solution of the equation

$$x^{(n)} + p_1(t)x^{(n-1)} = f(t),$$

$$|f(t)| \leq \mu, \quad |p_1(t)| \leq L_1 \quad (a \leq t \leq b),$$

$$x^{(n-1)}(c) = x^{(n-2)}(c) = \dots = x^{(k)}(c) = 0 \quad (a \leq c \leq b),$$

$$k = 0, 1, \dots, n-1,$$

then

$$|x^{(k)}(t)| \leq \frac{\mu}{L_1^{n-k}} \left\{ \exp[L_1|t-c|] - 1 - L_1|t-c| - \dots - \frac{[L_1|t-c|]^{n-k-1}}{(n-k-1)!} \right\}.$$

2. Let us note that inequalities (1) and (2) allow the coefficient  $L_1$  to take arbitrarily large values at the expense of the smallness of the remaining co-

efficients. The following theorem does not depend on the existing criteria (1-5), but it shows that the coefficient  $p_2(t)$  may assume arbitrarily large negative values in absolute value, owing to the smallness of the remaining coefficients.

**Theorem 3.** Let

$$|p_i(t)| \leq L_i, \quad i = 1, 3, 4, \dots, n,$$

$$-L_2 \leq p_2(t) \leq L_2^+ \quad (a \leq t \leq b).$$

The interval  $[a, b]$  of length  $b - a < 2 \min[h_1, h_2]$  is an interval of nonoscillation of the operator

$$L(x) = x^{(n)} + p_1 x^{(n-1)} + \dots + p_{nx},$$

if  $h_1, h_2$  are, respectively, the first positive roots of the equations

$$\sum_{k=3}^n \frac{L_k h_1^k}{(k-2) \left[\frac{k-3}{2}\right]! \left[\frac{k-2}{2}\right]!} = F_1(h_1), \quad \sum_{k=3}^n \frac{L_k h_2^k}{(k-2) \left[\frac{k-3}{2}\right]! \left[\frac{k-2}{2}\right]!} = F_2(h_2),$$

where

$$F_1(h) = \frac{L_2}{\left[\operatorname{ch} \eta h - \frac{L_1}{2\eta} \operatorname{sh} \eta h\right] \exp \frac{L_1}{2} h - 1}, \quad \eta = \sqrt{\frac{L_1^2}{4} + L_2};$$

$$F_2(h) = \begin{cases} \frac{L_2^+ \left[\operatorname{ch} \beta_1 h - \frac{L_1}{2\beta_1} \operatorname{sh} \beta_1 h\right] \exp \frac{L_1}{2} h}{1 - \left[\operatorname{ch} \beta_1 h - \frac{L_1}{2\beta_1} \operatorname{sh} \beta_1 h\right] \exp \frac{L_1}{2} h}, & \text{if } \beta_1^2 = \frac{L_1^2}{4} - L_2^+ > 0, \\ \frac{L_2^+ \left[\cos \beta_2 h - \frac{L_1}{2\beta_2} \sin \beta_2 h\right] \exp \frac{L_1}{2} h}{1 - \left[\cos \beta_2 h - \frac{L_1}{2\beta_2} \sin \beta_2 h\right] \exp \frac{L_1}{2} h}, & \text{if } \beta_2^2 = L_2^+ - \frac{L_1^2}{4} > 0, \\ \frac{L_2^+ \left[1 - \sqrt{L_2^+} h\right] \exp \sqrt{L_2^+} h}{1 - \left[1 - \sqrt{L_2^+} h\right] \exp \sqrt{L_2^+} h}, & \text{if } L_2^+ = \frac{L_1^2}{4}, \\ \frac{L_1^2}{\exp L_1 h - 1 - L_1 h}, & \text{if } L_2^+ = 0, \\ \frac{2}{h^2}, & \text{if } L_2^+ = L_1 = 0. \end{cases} \quad (3)$$

For  $n = 2$  the stated nonoscillation criterion coincides with the criterion, unimprovable in the characteristics  $L_1, L_2^+$  (6).

In the proof of Theorem 3 the following is used.

**Lemma.** Let the interval  $[a, b]$  be an interval of nonoscillation of the operator

$$L[x] = x'' + p_1 x' + p_2 x,$$

$$|p_1(t)| \leq L_1, \quad p_2(t) \leq L_2^+ \quad (a \leq t \leq b),$$

and let the function  $v(t)$  be a solution of the problem

$$Lv = 1, \quad v(a) = v(b) = 0.$$

Then

$$|v(t)| \leq \frac{1}{F_2[(b-a)/2]} \quad (a \leq t \leq b),$$

where  $F_2(h)$  is defined by the inequalities (3).

Bauman Higher Technical School

Received

15 V 1967

## CITED LITERATURE

- <sup>1</sup> D. Zh. Sansone, *Ordinary Differential Equations*, Moscow-Leningrad, 1953.
- <sup>2</sup> A. Yu. Levin, *Mathematical Collection*, 64 (106), 3, 396 (1964).
- <sup>3</sup> G. A. Bessmertnykh, A. Yu. Levin, *DAN*, 144, No. 3, 471 (1962).
- <sup>4</sup> A. Yu. Levin, *DAN*, 153, No. 6 (1963).
- <sup>5</sup> G. S. Zaitseva, *DAN*, 176, No. 4 (1967).
- <sup>6</sup> H. Erpheser, *Math. Zs.*, 61, No. 4 (1955).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*