

SOME ASYMPTOTIC FORMULAS FOR THE SPECTRA OF THE THIRD BOUNDARY-VALUE PROBLEM RELATED TO VARIATION OF THE FUNCTION ENTERING THE BOUNDARY CONDITION

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Abstract

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MATHEMATICS

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SOME ASYMPTOTIC FORMULAS FOR THE SPECTRA OF THE THIRD BOUNDARY-VALUE PROBLEM RELATED TO VARIATION OF THE FUNCTION ENTERING THE BOUNDARY CONDITION

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Let V be a bounded domain in three-dimensional space, and let S be its boundary. Further, let $\{\mu_k(\sigma)\}$ ($\mu_1(\sigma) \leq \mu_2(\sigma) \leq \dots$) be the eigenvalues of the operator Lu , generated by the operation $-\Delta u$ under the boundary condition

$$[\partial u / \partial n + \sigma u]_S = 0,$$

where $\partial u / \partial n$ is differentiation in the direction of the outward normal.

It is known (see ⁽¹⁾) that

$$\mu_n(\sigma) \sim (6\pi^2 v^{-1} n)^{2/3} \quad (n \rightarrow \infty) \quad (1)$$

(v is the volume of the domain V); from this it is not difficult to derive the relation

$$\sum_{\mu_k(\sigma) \leq x} \mu_k(\sigma_1) \sim \frac{v}{10\pi^2} x^{5/2} \quad (x \rightarrow +\infty),$$

where $\sigma(Q)$ and $\sigma_1(Q)$ ($Q \in S$) are two, generally speaking, different functions (the summation extends over all values of k for which $\mu_k(\sigma) \leq x$).

In this note the asymptotic behavior of the difference

$$\sum_{\mu_k(\sigma) \leq x} \mu_k(\sigma_2) - \sum_{\mu_k(\sigma) \leq x} \mu_k(\sigma_1);$$

is studied; namely, the equality

$$\sum_{\mu_k(\sigma) \leq x} [\mu_k(\sigma_2) - \mu_k(\sigma_1)] = \frac{1}{3\pi^2} \int_S [\sigma_2(Q) - \sigma_1(Q)] ds \cdot x^{3/2} + o(x^{3/2}) \quad (x \rightarrow +\infty). \quad (2)$$

is established.

Some of its generalizations are also given.

Formula (2) may be regarded as a generalization to the three-dimensional case of the well-known formula of I. M. Gelfand and B. M. Levitan for the difference of the traces of two Sturm-Liouville operators (see (2)).

Let Q be an arbitrary point of the surface S . We shall assume that some neighborhood of this point on the surface S can be given by the equation $z = z_Q(x, y)$ (the z -axis is directed along the normal to S at the point Q), and that the function $z_Q(x, y)$ everywhere in the disk $x^2 + y^2 < \rho_0^2$ (where ρ_0 does not depend on Q) has second partial derivatives satisfying a Lipschitz condition (with exponent equal to 1 and with a constant independent of Q).

Theorem 1. *If the function $\sigma(Q)$ satisfies a Lipschitz condition, then as $p \rightarrow +\infty$*

$$\sum_{k=1}^{\infty} \frac{1}{[\mu_k(\sigma) + p]^2} = \frac{v}{8\pi} \frac{1}{\sqrt{p}} + \frac{s}{16\pi} \frac{1}{p} + \frac{1}{24\pi} \int_S [h(Q) - 3\sigma(Q)] ds \frac{1}{p^{3/2}} + O\left(\frac{1}{p^2}\right), \quad (3)$$

where v and s are the volume of the domain V and the area of the surface S ; $h(Q)$ is the mean curvature of the surface S at the point Q .

Proof. Let $G(M, M_1, -\chi^2) = (4\pi r_{MM_1})^{-1} e^{-\chi r_{MM_1}} - g(M, M_1, \chi)$ be the Green's function of the expression $\Delta u - \chi^2 u$ (χ is a sufficiently large positive constant) for the domain V under the boundary condition $[\partial u / \partial n + \sigma u]_S = 0$. As $M_1 \rightarrow M$, from the equality

$$G(M, M_1, -\chi_1^2) - G(M, M_1, -\chi^2) = (\chi^2 - \chi_1^2) \sum_{k=1}^{\infty} \frac{u_k(M) u_k(M_1)}{[\mu_k(\sigma) + \chi_1^2][\mu_k(\sigma) + \chi^2]},$$

where $\{u_k(M)\}$ are the eigenfunctions of the operator Lu , and $\chi_1 \neq \chi$, it follows that

$$\frac{\chi_1 - \chi}{4\pi} + g(M, M, \chi_1) - g(M, M, \chi) = (\chi_1^2 - \chi^2) \sum_{k=1}^{\infty} \frac{u_k^2(M)}{[\mu_k(\sigma) + \chi_1^2][\mu_k(\sigma) + \chi^2]}.$$

Hence follows the formula

$$\sum_{k=1}^{\infty} \frac{1}{[\mu_k(\sigma) + \chi^2]^2} = \frac{v}{8\pi} \frac{1}{\chi} + \frac{1}{2\chi} \int_V g'_\chi(M, M, \chi) dv_M.$$

By means of a method close to that used in (3), it can be proved that

$$\begin{aligned} & \int_V g'_\chi(M, M, \chi) dv_M = \\ & = \frac{s}{8\pi} \frac{1}{\chi} + \frac{1}{12\pi} \int_S [h(Q) - 3\sigma(Q)] ds \cdot \frac{1}{\chi^2} + O\left(\frac{1}{\chi^3}\right) \quad (\chi \rightarrow +\infty). \end{aligned}$$

Combining the last two equalities and putting $\chi^2 = p$, we obtain (3).

Theorem 2. Let $\sigma(Q)$, $\sigma_1(Q)$, and $\sigma_2(Q)$ ($Q \in S$) be any three functions satisfying the Lipschitz condition. Then

$$\sum_{\mu_k(\sigma_2) \leq x} [\mu_k(\sigma_2) - \mu_k(\sigma_1)] = \frac{I}{3\pi^2} x^{3/2} + O(x^{3/2}) \quad (x \rightarrow +\infty), \quad (4)$$

where

$$I = \int_S [\sigma_2(Q) - \sigma_1(Q)] ds.$$

Proof. Suppose first that $\sigma_2(Q) \geq \sigma_1(Q)$. Introduce for consideration the nonnegative and nondecreasing function

$$\Phi(x) = \sum_{\mu_k(\sigma_2) \leq x} [\mu_k(\sigma_2) - \mu_k(\sigma_1)];$$

moreover, set

$$\psi(p) = \sum_{k=1}^{\infty} \frac{[\mu_k(\sigma_2) - \mu_k(\sigma_1)]^2 [\mu_k(\sigma_2) + 2\mu_k(\sigma_1) + 3p]}{[\mu_k(\sigma_2) + p]^3 [\mu_k(\sigma_1) + p]^2}.$$

We now use Theorem 1: let us write formula (4) first for the function $\sigma_1(Q)$, and then for the function $\sigma_2(Q)$; subtracting the second from the first equality, we find that for any $\alpha < \mu_1(\sigma_2)$

$$2 \int_{\alpha}^{+\infty} \frac{d\Phi(x)}{(x+p)^3} + \psi(p) = \frac{I}{8\pi} \frac{1}{p^{3/2}} + O\left(\frac{1}{p^2}\right) \quad (p \rightarrow +\infty).$$

Taking further into account that

$$\psi(p) = o \left[\int_{\alpha}^{+\infty} (x+p)^{-3} d\Phi(x) \right] \quad \text{as } p \rightarrow +\infty,$$

we have

$$\int_{\alpha}^{+\infty} \frac{d\Phi(x)}{(x+p)^3} \sim \frac{I}{16\pi} \frac{1}{p^{3/2}} \quad (p \rightarrow +\infty).$$

Applying the Hardy-Littlewood Tauberian theorem (see, for example, (4)), we find:

$$\Phi(x) = \sum_{\mu_k(\sigma_2) \leq x} [\mu_k(\sigma_2) - \mu_k(\sigma_1)] \sim \frac{I}{3\pi^2} x^{3/2} \quad (x \rightarrow +\infty).$$

Similarly, the asymptotic equality is established

$$\sum_{\mu_k(\sigma_1) \leq x} [\mu_k(\sigma_2) - \mu_k(\sigma_1)] \sim \frac{I}{3\pi^2} x^{3/2} \quad (x \rightarrow +\infty).$$

Taking further into account that

$$\sum_{\mu_k(\bar{\sigma}) \leq x} [\mu_k(\sigma_2) - \mu_k(\sigma_1)] \leq \sum_{\mu_k(\sigma) \leq x} \leq \sum_{\mu_k(\underline{\sigma}) \leq x},$$

where $\bar{\sigma}(Q) = \max\{\sigma(Q), \sigma_2(Q)\}$, $\underline{\sigma}(Q) = \min\{\sigma(Q), \sigma_1(Q)\}$, and that as $x \rightarrow +\infty$ the functions

$$\begin{aligned} & \sum_{\mu_k(\bar{\sigma}) \leq x} [\mu_k(\sigma_2) - \mu_k(\sigma_1)] = \\ &= \sum_{\mu_k(\bar{\sigma}) \leq x} [\mu_k(\bar{\sigma}) - \mu_k(\sigma_1)] - \sum_{\mu_k(\bar{\sigma}) \leq x} [\mu_k(\bar{\sigma}) - \mu_k(\sigma_2)], \\ & \sum_{\mu_k(\underline{\sigma}) \leq x} [\mu_k(\sigma_2) - \mu_k(\sigma_1)] \end{aligned}$$

are asymptotically equal to $\frac{I}{3\pi^2} x^{3/2}$, we obtain (4) for the special case $\sigma_2(Q) \geq \sigma_1(Q)$.

To prove the theorem in the general case, it is now enough to apply the equality

$$\sum_{\mu_k(\sigma) \leq x} [\mu_k(\sigma_2) - \mu_k(\sigma_1)] = \sum_{\mu_k(\sigma) \leq x} [\mu_k(\sigma_0) - \mu_k(\sigma_1)] - \sum_{\mu_k(\sigma) \leq x} [\mu_k(\sigma_0) - \mu_k(\sigma_2)],$$

where $\sigma_0(Q) = \max\{\sigma_1(Q), \sigma_2(Q)\}$, and to use the special case of this theorem considered above.

Taking (1) into account, we obtain

Corollary 1. As $n \rightarrow \infty$,

$$\sum_{k=1}^n [\mu_k(\sigma_2) - \mu_k(\sigma_1)] = \frac{2I}{v} n + o(n).$$

Remark. From Theorem 2 (or Corollary 1) it follows that if the integral

$$I = \int_S [\sigma_2(Q) - \sigma_1(Q)] ds \neq 0,$$

then for infinitely many values of the index k the difference $\mu_k(\sigma_2) - \mu_k(\sigma_1)$ has the same sign as the integral I .

With the aid of Theorem 2, the following can easily be proved.

Theorem 3. Let the functions $\sigma(Q)$, $\sigma_1(Q)$, $\sigma_2(Q)$ and the quantity I be the same as in Theorem 2. Suppose, further, that on the half-axis $[a, +\infty)$ there is given a sign-definite function $f(x)$, absolutely continuous on every interval $[a, b]$ ($b < +\infty$); in addition, assume that the expression $xf'(x)[f(x)]^{-1}$ is bounded almost everywhere and

$$\int^{+\infty} x^{1/2} f(x) dx = \infty.$$

Then, as $x \rightarrow +\infty$,

$$\sum_{a < \mu_k(\sigma) \leq x} f[\mu_k(\sigma)] [\mu_k(\sigma_2) - \mu_k(\sigma_1)] = \left(\frac{I}{2\pi^2} + o(1) \right) \int_a^x |t|^{1/2} f(t) dt.$$

Assuming that $f(x) = x^m$ ($m \geq -3/2$), we obtain

Corollary 2. As $x \rightarrow +\infty$,

$$\sum_{0 < \mu_k(\sigma) \leq x} \mu_k^m(\sigma) [\mu_k(\sigma_2) - \mu_k(\sigma_1)] = \begin{cases} \frac{I}{(2m+3)\pi^2} x^{m+3/2} + o(x^{m+3/2}), & (m > -3/2), \\ \frac{I}{2\pi^2} \ln x + o(\ln x), & (m = -3/2). \end{cases}$$

From Corollary 2 it is not difficult to derive the following assertions:

A. If $\mu_k(\sigma_1) \neq 0$, then as $n \rightarrow \infty$

$$\sum_{k=1}^n \frac{\mu_k(\sigma_2)}{\mu_k(\sigma_1)} = n + \left(\frac{6}{\pi^2 \nu} \right)^{1/3} I n^{1/3} + o(n^{1/3}).$$

B. If $\mu_1(\sigma_1) > 0$ and $\mu_1(\sigma_2) > 0$, then as $n \rightarrow \infty$

$$\sum_{k=1}^n [\mu_k^m(\sigma_2) - \mu_k^m(\sigma_1)] =$$

$$= \begin{cases} \frac{m}{(2m+1)\pi^2} \left(\frac{6\pi^2}{2} \right)^{(2m+1)/3} In^{(2m+1)/3} + o(n^{(2m+1)/3}), & (m > -1/2), \\ -\frac{I}{6\pi^2} \ln n + o(\ln n), & (m = -1/2). \end{cases}$$

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Note: Figure translations are in progress. See original paper for figures.

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