

A Parabolic-Type Equation with a Periodic Coefficient

G. Ya. Lyubarskii

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Abstract

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MATHEMATICS

G. Ya. LYUBARSKII

AN EQUATION OF PARABOLIC TYPE WITH A PERIODIC COEFFICIENT

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The subject of the present note is the Cauchy problem for the parabolic equation

$$\begin{aligned} \partial u / \partial t &= \partial^2 u / \partial x^2 + q(x, \tau)u, & \tau &= \nu t, \\ Au(x, t) &\equiv a_1 u'(0, t) + a_2 u(0, t) = 0, \\ Bu(x, t) &\equiv b_1 u'(1, t) + b_2 u(1, t) = 0, \\ \text{l. i. m. } u(x, t) &= \psi(x), \\ & \quad t \rightarrow 0 \end{aligned} \tag{1}$$

in the half-strip Π ($0 \leq x \leq 1$, $t > 0$). The numbers a_1, a_2, b_1 , and b_2 are real, with $a_1^2 + a_2^2 > 0$ and $b_1^2 + b_2^2 > 0$; $\psi(x)$ is an arbitrary quadratically integrable function; ν is a positive parameter. We shall be interested in those properties of problem (1) that it possesses for sufficiently large values of the parameter ν .

Regarding the function $q(x, \tau)$ the following is assumed. It is real and periodic with period 2π , expands in an absolutely convergent Fourier series

$$q(x, \tau) = \sum_{k=-\infty}^{\infty} q_k(x) e^{ik\tau}$$

with continuous coefficients, and has a quadratically integrable derivative with respect to t . In addition, it is assumed that there exists a sufficiently small positive number σ such that the series

$$\|q\|_{\sigma} = \max_{0 \leq x \leq 1} |q_0(x)| + \sum_{k=-\infty}^{\infty} |k|^{\sigma} \max_{0 \leq x \leq 1} |q_k(x)|$$

converges.

We shall agree to call a **Floquet solution** of problem (1) any function $u(x, t)$ satisfying the first three relations in (1) and representable in the form

$$u(x, t) = e^{\gamma t} v(x, t), \quad (2)$$

where $v(x, t)$ is a periodic function with period $T = 2\pi/\nu$.

Denote by Λ_μ ($0 \leq \mu \leq 1$) the totality of all periodic functions $y(x, t) = y(x, t + T)$ whose Fourier coefficients are continuous and satisfy the condition: the series

$$\|y\|_\mu = \max_{0 \leq x \leq 1} |y_0(x)| + \sum_{k=-\infty}^{\infty} |k|^\mu \max_{0 \leq x \leq 1} |y_k(x)| \quad (3)$$

converges. If $\|y\|_\mu$ is taken as the norm of the function $y(x, t)$, then Λ_μ becomes a Banach space.

Theorem 1. *If ν is greater than a certain number $\nu_m(\|q\|_0)$, depending only on the magnitude of $\|q\|_0$,*

$$\nu > \nu_m(\|q\|_0),$$

then problem (1) has at least m Floquet solutions $u_k(x, t)$ ($k = 1, 2, \dots, m$), belonging to Λ_1 .

Denote by L the operator

$$L = d^2/dx^2 + q_0(x),$$

considered on functions satisfying the boundary conditions $A\varphi = 0$, $B\varphi = 0$. Let $\varphi_k(x)$ and λ_k ($k = 1, 2, \dots$) be a complete set of eigenfunctions and eigenvalues of the operator L , with $\lambda_{k+1} < \lambda_k$. The eigenfunctions $\varphi_k(x)$ have norm equal to one in $\mathcal{L}_2(0, 1)$, and are uniformly bounded:

$$\int_0^1 \varphi_k^2(x) dx = 1, \quad \max_{0 \leq x \leq 1} |\varphi_k(x)| < \theta \quad (k = 1, 2, \dots).$$

Theorem 2. *If $\nu > \nu_m(\|q\|_0)$ and $\lambda_{m+1} < 0$, then the solution of problem (1) can be represented in the form*

$$u(x, t) = \sum_{k=1}^m c_k u_k(x, t) + w(x, t); \quad (4)$$

here c_k ($k = 1, 2, \dots, m$) are certain coefficients, determined below, and $w(x, t)$ is a function that can be estimated as follows:

$$\left| w(x, t) - \sum_{k=m+1}^{\infty} h_k e^{\lambda_k t} \varphi_k(x) \right| < \quad (5)$$

$$< 2\theta \|h\| \left[1 + 2(\alpha - \lambda_{m+1}) \sum_{k=m+2}^{\infty} (\lambda_{m+1} - \lambda_k)^{-1} \right]^{1/2} e^{\alpha t} F_m \left(Q_1 \sqrt{\frac{2t}{\alpha - \lambda_{m+1}}} \right);$$

h_k are the Fourier coefficients of the function

$$h(x) = \psi(x) - \sum_{k=1}^m c_k u_k(x, 0);$$

α is an arbitrary number from the interval $\lambda_{m+1} < \alpha \leq 0$;

$$q_1(x, \tau) \equiv q(x, \tau) - q_0(x), \quad Q_1 = \max_{x, \tau} |q_1(x, \tau)|; \quad (6)$$

$\|h\|$ denotes the norm of the function $h(x)$ in $\mathcal{L}_2(0, 1)$, and, finally,

$$F_m(z) \equiv \sum_{n=m+1}^{\infty} \frac{z^n}{\sqrt{n!}}.$$

The function $w(x, t)$ has a continuous derivative with respect to t in the half-strip Π .

Let us note that the right-hand side of inequality (5) for $\alpha = 0$ decreases as the index m grows proportionally to $|\lambda_m|^{-(m+1)/2}$ uniformly in t ($t \geq 0$). On the other hand, with a suitable choice of the parameter α , the right-hand side does not exceed the quantity

$$M e^{(\lambda_{m+1} + 2Q_1^2 + \varepsilon)t}$$

for all $t \geq 0$, if $\varepsilon > 0$ is any positive number and M is a sufficiently large number.

Denote by $z_k(x, t) \in \Lambda_1$ ($k = 1, 2, \dots, m$) the Floquet solutions of problem (1) in which the function $q(x, t)$ is replaced by the function $q(x, -t)$. The functions $z_k(x, t)$ may be renumbered and normalized so that

$$\int_0^1 u_i(x, t) z_k(x, -t) dx = \delta_{ik} \quad (i, k = 1, 2, \dots, m).$$

Theorem 3. The coefficients c_k ($k = 1, 2, \dots, m$) appearing in relations (4) and (5) can be computed by the formula

$$c_k = \int_0^1 \psi(x) z_k(x, 0) dx.$$

The following two theorems make it possible to find Floquet solutions $u_k(x, t)$ and the function $w(x, t)$.

Theorem 4. We represent the Floquet solution $u_k(x, t)$ ($k = 1, 2, \dots, m$) in the form (2). The function $v(x, t)$ is the limit in the norm Λ_1 of the sequence $v_n(x, t)$, defined as follows:

1. $v_0(x, t) \equiv \varphi_k(x)$.
2. If the function $v_n(x, t)$ ($n = 0, 1, \dots$) has already been defined, then the function $v_{n+1}(x, t)$ is found as the solution in Λ_1 of the problem

$$\partial v_{n+1} / \partial t = (L - \lambda_k) v_{n+1} + [q_1(x, t) - \gamma_n + \lambda_k] v_n, \quad (7)$$

$$Av_{n+1} = 0, \quad Bv_{n+1} = 0, \quad A^+ v_{n+1,0} \equiv a_1 v_{n+1,0}(0) - a_2 v'_{n+1,0}(0) = A^+ \varphi_k(x),$$

where

$$\gamma_n = \lambda_k + \int_0^1 \varphi_k(x) (q_1(x, t) v_n(x, t))_0 dx / \int_0^1 \varphi_k(x) v_{n,0}(x) dx. \quad (8)$$

All the numbers γ_n determined in this way are finite, and the sequence of these numbers tends to γ . The rate of approximation of the function $v(x, t)$ by the functions $v_n(x, t)$ is characterized by the following estimates:

$$|v_{n+1,0}(x) - v_{n,0}(x)| < N_0 \left(\frac{a_0}{\nu} \right)^{1+[n/2]},$$

$$\|v_{n+1,1}(x, t) - v_{n,1}(x, t)\|_0 < N_1 \left(\frac{a_1}{\nu} \right)^{1+[\frac{n+1}{2}]} \quad (n = 0, 1, \dots). \quad (9)$$

Here N_0, N_1, a_0 , and a_1 are some constants independent of ν .

The indices zero and one in equalities (7), (8), and (9) have the same meaning as in definition (6).

Theorem 5. The function $w(x, t)$ is equal to

$$w(x, t) = w_0(x, t) + \sum_{n=1}^{\infty} w_n(x, t); \quad (10)$$

the series on the right-hand side converges uniformly in Π , and the functions $w_n(x, t)$ ($n = 0, 1, \dots$) are defined as follows:

$$w_n(x, t) = \sum_{k=1}^m c_k^{(n)}(t) \varphi_k(x) + \xi_n(x, t) \quad (n = 0, 1, \dots), \quad (11)$$

$$\xi_0(x, t) = \sum_{k=m+1}^{\infty} h_k e^{\lambda_k t} \varphi_k(x), \quad \xi_n(x, t) = \sum_{k=m+1}^{\infty} \varphi_k(x) \int_0^t e^{\lambda_k(t-s)} f_{n-1,k}(s) ds, \\ (n = 1, 2, \dots), \quad (12)$$

$f_{n,k}(s)$ are the Fourier coefficients of the function $q_1(x, \nu s) w_n(x, s)$ with respect to the functions $\varphi_k(x)$ ($k = 1, 2, \dots$). The coefficients $c_k^{(n)}(t)$ are determined from the conditions

$$\int_0^1 w_n(x, t) z_k(x, -t) dx = 0 \quad (t \geq 0, k = 1, 2, \dots, m; n = 0, 1, \dots). \quad (13)$$

The terms of the series (10) can be estimated as follows:

$$|w_n(x, t)| \leq 2\theta \|h\| \left[1 + 2(a - \lambda_{m+1}) \sum_{k=m+2}^{\infty} (\lambda_{m+1} - \lambda_k)^{-1} \right]^{1/2} \times \\ \times \frac{e^{at}}{\sqrt{n!}} \left[\frac{2Q_1^2 t}{\alpha - \lambda_{m+1}} \right]^{n/2}, \quad (14)$$

where a is any number from the interval $(\lambda_{m+1} < a \leq 0)$.

The function $w(x, t)$ is continuously differentiable with respect to t in Π . Its derivative can be obtained by termwise differentiation of the series (10). The estimate

$$|\dot{w}_n(x, t)| \leq 2\sigma_n(t_0)(t/t_0)^{3/4n}(1 - t_0/t)^{-1/3} \quad (n = 1, 2, \dots),$$

holds, where $t_0 < t$ is an arbitrary positive number, and $\sigma_n(t_0)$ is a collection of functions such that

$$\lim_{n \rightarrow \infty} \sqrt[n]{\sigma_n(t_0)} = 0.$$

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Note: Figure translations are in progress. See original paper for figures.

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