

# FINITENESS OF THE NUMBER OF RATIONAL AND ALGEBRAIC POINTS ON SOME TRANSCENDENTAL CURVES

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**Abstract**

**Full Text**

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MATHEMATICS

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## FINITENESS OF THE NUMBER OF RATIONAL AND ALGEBRAIC POINTS ON SOME TRANSCENDENTAL CURVES

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Studying the Diophantine properties of the values of entire functions, C. Siegel<sup>(1)</sup> introduced a class of functions  $\varphi(z)$ , which he called *E*-functions:

$$\varphi(z) = \sum_{\nu=0}^{\infty} c_{\nu} \frac{z^{\nu}}{\nu!}; \quad (1)$$

- (a) the numbers  $c_{\nu}$  are algebraic and belong to a field  $K$  of finite degree over the field of rational numbers  $Q$ ; (b) for every  $\varepsilon > 0$ ,  $\overline{[c_{\nu}]} < c(\varepsilon)(\nu!)^{\varepsilon}$ , where  $\overline{[x]}$  for  $x \in K$  denotes the maximum of the moduli of all quantities conjugate to  $x$  in  $K$ ; (c) if  $q_m$  is the least natural number for which  $c_{\nu} q_m$  ( $\nu = 0, 1, \dots, m$ ) are integers, then  $q_m < c(\varepsilon)(m!)^{\varepsilon}$  for every  $\varepsilon > 0$ .

Siegel established<sup>(2)</sup> the transcendence and algebraic independence of values at algebraic points of *E*-functions satisfying a system of linear homogeneous differential equations with polynomial coefficients, under the additional, difficult-to-verify condition of “normality” of the system of these equations. Siegel’s method was substantially developed by A. B. Shidlovskii<sup>(4–6)</sup>, who studied the values of *E*-functions satisfying a system of nonhomogeneous linear differential equations with polynomial coefficients, replacing Siegel’s “normality” condition by the obvious necessary condition of algebraic independence of the system of *E*-functions over the field of rational functions, thereby obtaining a final result. The Siegel-Shidlovskii theory describes the arithmetic nature of values at algebraic points of a great many entire functions ( $\sin z$ ,  $\cos z$ ,  $e^z$ , the Bessel function  $J_0(z)$ , the confluent hypergeometric function, the incomplete gamma function, Fresnel integrals, etc.). Meanwhile, it is known that any weakening of the conditions defining *E*-functions completely deprives one of the possibility of carrying out arguments by Siegel’s method. For example, the function

$$\varphi_{\lambda}(z) = \sum_{n=0}^{\infty} \frac{z^n}{(\lambda+1) \dots (\lambda+n)}, \quad \lambda \neq -1, -2, \dots, \quad (2)$$

which satisfies the differential equation  $zy' + (\lambda - z)y - \lambda = 0$  and, for rational  $\lambda$ , belongs to the class of  $E$ -functions, for quadratic irrational  $\lambda$  does not belong to this class, and the arithmetic nature of its values for such  $\lambda$  is not decided by Siegel's method. The fact that  $\varphi_\lambda(z)$ , for rational  $\lambda \neq -1, -2, \dots$ , belongs to the class of  $E$ -functions follows from a general theorem of Siegel <sup>(2)</sup>: the functions

$$\varphi(z) = \sum_{n=0}^{\infty} \frac{[\alpha_1, n] \dots [\alpha_l, n]}{[\beta_1, n] \dots [\beta_m, n]} z^{tn}, \quad t = m - l \geq 1, \quad (3)$$

$$[\alpha, 0] = 1, \quad [\alpha, n] = \alpha(\alpha + 1) \dots (\alpha + n - 1) \quad (n = 1, 2, \dots)$$

with rational  $\alpha_j, \beta_j \neq 0, -1, -2, \dots$  belong to the class of  $E$ -functions and satisfy linear differential equations with polynomial coefficients. Siegel called these functions "hypergeometric."

Shidlovskii <sup>(6,7)</sup> has repeatedly drawn attention to the importance of investigating the Diophantine properties of entire functions that do not belong to the class of Siegel  $E$ -functions. In particular, quite recently Shidlovskii himself <sup>(8)</sup> established the irrationality of all, with the exception of a finite number, of the values of the functions  $\varphi_\lambda(z)$ ,  $K_\lambda(z)$ ,  $K'_\lambda(z)$ , where

$$K_\lambda(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(\lambda + 1) \dots (\lambda + n)} \left(\frac{z}{2}\right)^{2n}, \quad \lambda \neq -1, -2, \dots,$$

for algebraic  $\lambda$ ,  $z \in \mathbf{K}$ ,  $\mathbf{K}$  being a fixed field of algebraic numbers.

We introduce the class of  $E^*$ -functions as the class of functions of the form (1), weakening Siegel's conditions (b) and (c) (but assuming that (a) is satisfied): (b\*) there exists a number  $a$ ,  $0 \leq a < 1$ , for which  $|c_\nu| < c(\varepsilon)(\nu!)^{a+\varepsilon}$ , where  $\varepsilon > 0$  is arbitrary; (c\*) for fixed  $n$ , let  $q_{nm}$  be the smallest natural number  $q$  for which all the numbers  $qc_{\nu_1}c_{\nu_2} \dots c_{\nu_n}$ ,  $\nu_1 + \nu_2 + \dots + \nu_n \leq m$ , are integers; there exists a sequence of numbers  $\beta_n$ ,  $0 \leq \beta_n = o(\sqrt{n})$  as  $n \rightarrow \infty$ , for which  $q_{nm} < c(n, \varepsilon)(m!)^{\beta_n + \varepsilon}$  for any  $\varepsilon > 0$ .

It is clear that when  $a = \beta_n = 0$  ( $n = 1, 2, \dots$ ) the function  $\varphi(z)$  will be an  $E$ -function, so that the class of  $E^*$ -functions contains Siegel's class of  $E$ -functions. It turns out that, applying the original Thue-Siegel idea for the construction of approximating forms, and then making a direct passage to numerical values without the construction and investigation of a system of linearly independent forms required in Siegel's method, one can obtain some results on the Diophantine properties of values of  $E^*$ -functions satisfying linear differential equations with polynomial coefficients. By this method, at the present stage of its development, it is not possible to establish the transcendence of values of  $E^*$ -functions, although assertions of the following type are proved:

(A) For any natural number  $d$  and any field of algebraic numbers  $\mathbf{K}_1$  of finite degree over  $Q$ , there exists only a finite number of  $z \in \mathbf{K}_1$  for which the function  $\varphi(z)$  assumes algebraic values of degree at most  $d$  over  $Q$ . In particular, on the curve  $y = \varphi(z)$  there lie no more than finitely many points  $(y, z) \in \mathbf{K}_1 \times \mathbf{K}_1$ .

**Theorem 1.** If a transcendental  $E^*$ -function  $\varphi(z)$  satisfies the differential equation

$$p(z)y' + q(z)y + r(z) = 0, \quad (4)$$

where  $p(z)$ ,  $q(z)$ ,  $r(z)$  are polynomials with algebraic coefficients, then assertion (A) holds for  $\varphi(z)$ .

**Theorem 2.** Every transcendental function of the form

$$\varphi(z) = e^{Q(z)} \left( \int_0^z R(x) e^{-Q(x)} dx + c \right),$$

where  $Q(z)$ ,  $R(z)$  are polynomials with algebraic coefficients and  $c$  is an arbitrary complex number, satisfies condition (A).

**Theorem 3.** Siegel hypergeometric functions (3) with algebraic parameters  $\alpha_i, \beta_j \neq 0, -1, -2, \dots$  belong to the class of  $E^*$ -functions. In particular, this is true for the functions (2) with algebraic  $\lambda$ , so that (A) holds for them.

Theorem 1 is proved according to the following scheme. Applying Lemma II from (3), p. 67, one can, for any  $h > h(n, \varepsilon)$ , construct a system of polynomials  $P_j(z)$  with integer coefficients from  $\mathbf{K}$ , of degree at most  $h$ , satisfying the conditions:

$$0 \neq H = \max_{0 \leq j \leq n} |\overline{P_j(z)}| < H_0^{2(\alpha + \beta n) + \varepsilon}, \quad H_0 = h^h, \quad (5)$$

function

$$f(z) = \sum_{j=0}^n P_j(z) \varphi^j(z) \quad (6)$$

at  $z = 0$  has a zero of order  $m \geq h[\sqrt{n}]$ . Suppose that the function  $f(z)$  has, at the points  $z = x_i \neq 0$ , zeros of orders  $m_i$ , respectively ( $i = 1, 2, \dots, k$ ),  $m_1 = \max_{(i)} m_i$ . Applying the integral representation

$$f^{(m_1)}(x_1) = \frac{m_1!}{(2\pi i)^2} \int_{|z-x_1|=1} \frac{dz}{(z-x_1)^{m_1+1}} \times$$

$$\times \int_{|\xi|=R} \left(\frac{z}{\xi}\right)^m \left(\frac{z-x_1}{\xi-x_1}\right)^{m_1} \dots \left(\frac{z-x_k}{\xi-x_k}\right)^{m_k} \frac{f(\xi)}{\xi-z} d\xi$$

and choosing a suitable  $R > \max_{(i)} |x_i| = \rho$ , we find

$$0 \neq |f^{(m_1)}(x_1)| < H_0^{1-([\sqrt{n}]-1)/v+\varepsilon} m_1^{-(k/v-1)m_1+\varepsilon}, \quad (7)$$

where  $v = 1/(1-\alpha)$  is the order of the function  $\varphi(z)$ ,  $h > h(n, \varepsilon, \rho)$ . The polynomials  $p, q, r$  in equation (4) may be regarded as polynomials with integral coefficients over  $K$ , of degrees, say,  $\leq s$ ,  $|p| + |q| + |r| \leq a$ . Define the polynomials  $P_{jl} = P_{jl}(z)$  by the equalities  $P_{j0} = P_j$  ( $j = 0, 1, \dots, n$ ), where the  $P_j$  are determined by conditions (5), (6), and  $P_{jl}$  ( $l = 1, 2, \dots$ ) by the equalities

$$P_{0l+1} = pP'_{0l} - rP_{1l},$$

$$P_{jl+1} = pP'_{jl} - jqP_{jl} - (j+1)rP_{j+1,l} \quad (j = 1, 2, \dots, n-1),$$

$$P_{nl+1} = pP'_{nl} - nqP_{nl}.$$

Then

$$\max_{0 \leq j \leq n} |\overline{P_{jl}}| < a^{lH} (h + sl + 2n + 1)^l. \quad (8)$$

Putting  $f_0(z) = f(z)$ ,  $f_{l+1}(z) = p(z)f'_l(z)$ , we find

$$f_l(z) = \sum_{j=0}^n P_{jl}(z) \varphi^j(z) \quad (l = 0, 1, \dots), \quad f_{m_1}(x_1) = p^{m_1}(x_1) f^{(m_1)}(x_1).$$

If we assume that the numbers  $x_1, x_2, \dots, x_k$  are algebraic and belong to the field  $K_1$ , then, in view of (5), (7), (8), we can establish (A), if  $n$  is taken sufficiently large and it is assumed that  $k$  can be sufficiently large.

Passing from equation (4) to the system of equations

$$y'_l = \sum_{j=1}^m Q_{jl} y_j + Q_l \quad (l = 1, 2, \dots, m), \quad (9)$$

where  $Q_{jl}, Q_l$  are rational functions with algebraic coefficients, one can prove, for a system of  $E^*$ -functions, assertions of the following type:

- (B) For any natural number  $d$  and any field of algebraic numbers  $K_1$  of finite degree over  $Q$ , there exists only a finite number of  $z \in K_1$  for which the functions  $\varphi_1(z), \dots, \varphi_m(z)$  simultaneously take algebraic values of degrees not exceeding  $d$  over  $Q$ . In particular, on the curve  $\Gamma = (\varphi_1(z), \dots, \varphi_m(z))$  there lie no more than a finite number of points from  $K_1^m$  for  $z \in K_1$ .

**Theorem 4.** If the  $E^*$ -functions  $\varphi_1(z), \dots, \varphi_m(z)$  satisfy the system of equations (9) and are algebraically independent over the field of rational functions  $C(z)$ ,  $C$  being the field of complex numbers, then they satisfy condition (B).

**Theorem 5.** If all the coefficients of equation (9) are polynomials, then for any system of functions  $\varphi_1(z), \dots, \varphi_m(z)$  satisfying (9) and algebraically independent over  $C(z)$ , assertion (B) is valid.

The proof of Theorem 4 is carried out according to the same scheme as the proof of Theorem 1, and contains only technical complications. The proof of Theorem 2 and of the more general Theorem 5 is based on the following initial remarks. If the functions  $\varphi_1(z), \dots, \varphi_m(z)$  at an algebraic point  $\alpha$  take algebraic values simultaneously, then, by virtue of (9), we can establish that the functions  $\varphi_1(z + \alpha), \dots, \varphi_m(z + \alpha)$  belong to the class  $E^*$ . For the proof of (6\*) it should be noted that equation (9) is satisfied only by entire functions of finite order (by the condition of Theorem 5,  $Q_{jl}, Q_l$  are polynomials), and the proof of (B\*) is obtained by successive differentiation of (9), with simple estimates. Then we apply Theorem 4.

Finally, both the membership of Siegel hypergeometric functions with algebraic parameters in the class of  $E^*$ -functions, and the nonmembership of the function (2) with quadratic irrationality in the class of  $E^*$ -functions, are proved by the application of  $p$ -adic analysis.

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*Note: Figure translations are in progress. See original paper for figures.*

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