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Abstract

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MATHEMATICS

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ON ONE CLASS OF DEGENERATING QUASI-LINEAR EQUATIONS

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Let two differential operators be given:

$$L_1 = \frac{1}{2} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i}; \quad L_2^v = \sum_{i=1}^n B_i(t, x, v) \frac{\partial}{\partial x_i}.$$

It is assumed that $\sum_{i,j=1}^n a_{ij}(x) \lambda_i \lambda_j \geq 0$; the matrix $\{a_{ij}(x)\}$ is representable in the form $\{a_{ij}(x)\} = \sigma(x) \sigma^*(x)$; the functions $\sigma_{ij}(x)$, $b_i(x)$, $B_i(t, x, v)$ satisfy a Lipschitz condition in all arguments and are bounded for $t \in [0, \infty)$, $x \in D \equiv R^n$, $v \in [a, b]$. We shall say that the operator L_2^v is **subordinate** to the operator L_1 ($L_2^v \prec L_1$) in the domain $D_1 = [0, \infty) \times D \times [a, b]$, if the system of linear algebraic equations

$$\sum_{j=1}^n \sigma_{ij}(x) \varphi_j(t, x, v) = B_i(t, x, v),$$

$i = 1, \dots, n$, has a solution $\{\varphi_i(t, x, v)\}$, all elements of which are bounded continuous functions in D_1 .

The definition just given formally depends on the matrix $\sigma(x)$, which is determined nonuniquely by the condition $\sigma(x) \sigma^*(x) = \{a_{ij}(x)\}$. It is easy to verify that, if $\sigma_1(x) \sigma_1^*(x) = \sigma_2(x) \sigma_2^*(x) = \{a_{ij}(x)\}$, then there is an orthogonal matrix Q such that $\sigma_2(x) Q(x) = \sigma_1(x)$. If $\sigma_2 \varphi = B$, then $\sigma_1(Q^{-1} \varphi) = B$. Therefore the definition of subordination of operators does not depend on the choice of the matrix $\sigma(x)$.

Everywhere in this note we assume that the functions $\varphi_i(t, x, v)$ satisfy a Lipschitz condition in the variables x and v and are continuous in t . It is obvious that if L_1 is uniformly elliptic in D_1 , then $L_2^v \prec L_1$.

Consider the Cauchy problem

$$\partial u / \partial t = L_1 u(t, x) + L_2^u u, \quad u(0, x) = f(x). \quad (1)$$

We assume that $L_2^v \prec L_1$ in $D_1 = [0, \infty) \times R^n \times (\inf f(x), \sup f(x)]$. Suppose that problem (1) has a classical solution. To the operator $L_u = L_1 + L_2^u$ (u being the solution of the Cauchy problem) there corresponds a certain Markov process $\tilde{X} = \{\tilde{x}_t, \tilde{P}_{t,x}\}$ (see ⁽¹⁾). We denote by $X = \{x_t, P_x\}$ the Markov process governed by the operator L_1 . If $L_2^u \prec L_1$, then the measures in the space of functions μ and $\tilde{\mu}$, induced by the processes X and $\tilde{X}$, are absolutely continuous with respect to each other (see ^(2,3)) and

$$\frac{d\tilde{\mu}}{d\mu} = \exp \left\{ \int_0^t \varphi(t-s, x_s, u(t-s, x_s)) d\xi_s - \frac{1}{2} \int_0^t \sum_{i=1}^n \varphi_i^2(t-s, x_s, u(t-s, x_s)) ds \right\},$$

where ξ_t is an n -dimensional Wiener process "tangent" * to X . Using the expression for the density $d\tilde{\mu}/d\mu$, for $u(t, x)$ we obtain the expression

$$u(t, x) = \tilde{M}_{t,x} f(\tilde{x}_t) = M_x f(x_t) \exp \left\{ \int_0^t \varphi(t-s, x_s, u(t-s, x_s)) d\xi_s - \frac{1}{2} \int_0^t \sum_{i=1}^n \varphi_i^2(t-s, x_s, u(t-s, x_s)) ds \right\}. \quad (2)$$

A function $u(t, x)$ satisfying, for all (t, x) , identity (2) will be called a **generalized solution of the Cauchy problem**.

If the matrix $\{\sigma_{ij}(x)\}$ degenerates, then the functions $\{\varphi_i(t, x, v)\}$ are not determined uniquely. Nevertheless, the definition of a generalized solution is correct. The point is that a function $u(t, x)$ satisfying (2) for some $\{\varphi_i\}$ satisfies identity (2) also for other $\{\varphi_i\}$ that solve the equation $\{\sigma_{ij}(x)\}\{\varphi_i\} = \{B_i(t, x, v)\}$. The generalized solution does not depend on the choice of the matrix $\sigma(x)$.

This generalized solution can be defined in several ways. Let $A_{v(t,x)}$ be the infinitesimal operator of the process constructed from the operator $L_{v(t,x)} = L_1 + L_2^{v(t,x)}$. A function $u(t, x)$ satisfies identity (2) if and only if it is a solution of the problem

$$\partial u(t, x) / \partial t = A_u u, \quad u(0, x) = f(x).$$

It is easily verified that the generalized solution assumes the initial conditions, and if it is sufficiently smooth, then equation (1) is satisfied. As follows from Theorem 2, the generalized solution introduced here is also a solution in the small-parameter sense.

Theorem 1. *Suppose $L_2^v \prec L_1$ in $D_1 = [0, \infty) \times R^n \times [\inf_{R^n} f(x), \sup_{R^n} f(x)]$. Then problem (1) has a unique generalized solution in the class of bounded measurable functions. If the initial function is continuous, then the generalized solution is also continuous.*

The proof of this theorem is obtained by the method of successive approximations. It turns out that there exists a $t_0 > 0$, depending only on

$$\max_{i,j,t,x,v} \{ |a_{ij}(x)|, |b_i(x)|, |\sigma_{ij}(x)|, |\varphi_i(t, x, v)|, |B_i(t, x, v)| \}$$

and their derivatives with respect to x and v , as well as on $\max |f(x)|$, such that for $t < t_0$ the mapping

$$w(t, x) = F[v(t, x)] = M_x f(x_t) \exp \left\{ \int_0^t \varphi(t-s, x_s, v(t-s, x_s)) d\xi_s - \frac{1}{2} \int_0^t \sum_1^n \varphi_i^2(t-s, x_s, v) ds \right\}$$

will be a contraction. Since

$$\max_{x \in R^n} |u(t, x)| < \max_{x \in R^n} |f(x)|,$$

by steps of size t_0 one can reach any t , and consequently a solution exists globally.

There is reason to believe that the requirement $L_2^v \prec L_1$ is close to necessary for the existence of a continuous solution for every continuous initial function. In any case, if $\varphi(t, x, v) = \infty$ on a set open in t and x , and on this set $B_i(t, x, v)$ depend essentially on v , then for some $f(x) \in C$ the solution must be discontinuous.

Denote by $u^\varepsilon(t, x)$ the solution of problem (1) for the operator $L_u^\varepsilon = L_u + \varepsilon \tilde{L}$, where $\tilde{L}$ is a linear uniformly elliptic second-order operator with continuously differentiable bounded coefficients.

Theorem 2. *Suppose the conditions of Theorem 1 are satisfied and $f(x)$ satisfies the Lipschitz condition. Then*

$$u(t, x) = \lim_{\varepsilon \rightarrow 0} u^\varepsilon(t, x)$$

uniformly on every compact set.

* This means that the trajectories of the process X satisfy stochastic equations in which the process ξ_t is taken as the principal Wiener process.

The following theorem shows that the condition $L_2^v < L_1$ ensures the smoothness of a generalized solution, provided only that the coefficients and the initial function are sufficiently smooth.

Theorem 3. Let $L_2^v < L_1$ in

$$D_1 = [0, \infty) \times R^n \times \left[\min_{R^n} f(x), \max_{R^n} f(x) \right].$$

Suppose that all coefficients of the operator L_2 , the initial function, and the functions $\varphi_i(t, x, v)$ belong to the class

$$C^{(k)}(R^n \times [\min f(x), \max f(x)]).$$

Then the first k derivatives of the function $u(t, x)$ with respect to x admit an a priori estimate depending only on the maximum of the moduli of the coefficients of the equations, the functions $\varphi_i(t, x, v)$, the initial function, and their first k partial derivatives.

To obtain the required estimate, we differentiate (2) with respect to x (for simplicity, let $n = 1$). Then for the function

$$v(t) = \max_{x \in R^1} \left| \frac{\partial u}{\partial x}(t, x) \right|$$

we obtain an inequality of the type

$$v^4(t) \leq A_1 + A_2 \int_0^t v^4(s) ds,$$

valid on every interval $[0, T]$; the constants A_1, A_2 depend on T . From the displayed inequality it follows that

$$v^4(t) \leq A_1 e^{A_2 t}.$$

We now turn to the first boundary-value problem

$$\begin{aligned} L_u u(t, x) &= \partial u / \partial t \quad \text{for } (t, x) \in D \times (0, T] = \Pi, \\ u(0, x) &= f(x), \quad u(t, x)|_{(t, x) \in \widetilde{\partial \Pi}} = \psi(t, x). \end{aligned} \quad (3)$$

Here $\psi(t, x)$ is a continuous function defined on a subset $\widetilde{\partial \Pi}$ of the boundary $\partial \Pi$ of the cylinder Π . In order that problem (3) have a unique solution, the

boundary conditions are prescribed on the set of regular points of the boundary $\partial\Pi$.

The notion of regularity in the present case requires additional explanation. If one considers problem (3) for the linear equation $\partial u/\partial t = L_1 u$, then a point $(t, x) \in \partial\Pi$ is called regular if, for every continuous boundary function $\psi(t, x)$, the solution of the problem (generalized, if there is no classical one) is continuous at the point (t, x) . Generally speaking, the regularity of boundary points for degenerate quasilinear equations depends on the boundary function. However, in our case ($L_2^v < L_1$) it turns out that one should regard as regular those and only those points that are regular for the operator $\partial/\partial t = L_1$. Thus, let $\widetilde{\partial\Pi}$ be the set of points of the boundary of the cylinder Π that are regular for the equation $\partial u/\partial t = L_1 u$, and let $\psi(t, x)$ be a continuous function defined on $\widetilde{\partial\Pi} \cup D$, $\psi(0, x) = f(x)$.

By a generalized solution of problem (3) we shall mean a function satisfying the relation

$$u(t, x) = M_x \psi(\tau_t, x_{\tau_t}) \exp \left\{ \int_0^{\tau_t} \varphi(t-s, x_s, u(t-s, x_s)) d\xi_s - \frac{1}{2} \int_0^t \sum_{i=1}^n \varphi_i^2(t-s, x_s, u(t-s, x_s)) x_s ds \right\},$$

where

$$\tau_t = \min\{t, \inf\{t : x_t \in D\}\}.$$

As in the case of the Cauchy problem, this definition is correct.

Theorem 4. Let $L_2^v < L_1$; let $\widetilde{\partial\Pi}$ be the set of boundary points regular for the equation $\partial u/\partial t = L_1 u$. Then problem (3) has a unique generalized solution $u(t, x)$ in the class of bounded measurable functions. The function $u(t, x)$ assumes the boundary values at the points of the set $\widetilde{\partial\Pi} \cup D$.

The proof of this theorem is analogous to the proof of Theorem 1.

Theorem 5. If the conditions of Theorem 4 are satisfied, $\psi(t, x)$ is continuous on $\widetilde{\partial\Pi} \cup D$, and, moreover, all attainable boundary points are uniformly regular (see (4)), then the generalized solution is continuous.

Under the conditions of Theorem 5 there is an analogue of Theorem 2. It is also possible, under natural restrictions, to obtain an a priori estimate for the first derivatives.

Theorem 6. Suppose the conditions of Theorem 4 are satisfied and the operator L_1 degenerates only on the boundary. Then the generalized solution is twice continuously differentiable inside the domain D with respect to the spatial variables and differentiable with respect to t , i.e. it is classical.

We now consider a mixed problem with a reflection condition. Let $l(x)$ be a field on the boundary Γ of the domain D , and let $\det\{a_{ij}(x)\} \neq 0$ for $x \in \Gamma$. By a generalized solution of the problem

$$\frac{\partial u}{\partial t} = L_u u + f(x), \quad \frac{\partial u}{\partial e} \Big|_{\Gamma \times [0, T]} = 0, \quad u(0, x) = f(x) \quad (4)$$

we shall mean a function $u(t, x)$ satisfying the relation

$$u(t, x) = M_x \int_0^t f(\hat{x}_s) ds \exp \left\{ \int_0^t \varphi(t-s, \hat{x}_s, u(t-s, \hat{x}_s)) d\xi_s - \frac{1}{2} \int_0^t \sum \varphi_i^2(t-s, \hat{x}_s, u) ds \right\},$$

where $\hat{x}_t$ is a process in the domain D , governed by the operator L_1 with reflection in the direction $l(x)$ on Γ (see (5)). For problem (4) there are also theorems on existence and uniqueness of a continuous solution.

A similar approach makes it possible to obtain existence and uniqueness theorems also for elliptic equations under the assumption that the domain in which the problem is considered is small in at least one direction in which the operator does not degenerate or the coefficient of the first derivative does not vanish.

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Note: Figure translations are in progress. See original paper for figures.

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