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Abstract

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MATHEMATICS

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ON THE THEORY OF ORLICZ SPACES GENERATED BY VARIABLE N -FUNCTIONS

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1°. In the present article we describe some properties of Orlicz classes $K_M(G)$ and Orlicz spaces $L_M(G)$, which are constructed with the aid of so-called variable N -functions $M(x, w)$. The theorems proved here are a generalization of the corresponding results for Orlicz spaces $L_M(G)$ in the case when $\text{mes } G < \infty$ and $M(x, w) = M(w)$ (see ⁽¹⁾), as well as of the results of the note ⁽³⁾; here the known Δ_2 -condition (⁽¹⁾, p. 35) for N -functions and the $\widetilde{\Delta}_2$ -condition introduced in ⁽³⁾ are particular cases of the more general Δ'_2 -condition (see Definition 2). We note that a complete exposition of the theory of N -functions and Orlicz spaces in the case when $M(x, w) = M(w)$, and G is a closed set of finite measure situated in n -dimensional Euclidean space, is available in ⁽¹⁾ (there one also finds detailed references to the literature on this question). The case $\text{mes } G = \infty$ and $M(x, w) = M(w)$ was studied in the works ⁽⁴⁻⁶⁾.

2°. Everywhere in what follows, by G we denote some space of points x with a σ -finite measure. We shall call G a **space with a continuous measure** if, for every set $\mathcal{E} \subset G$ of finite measure and every $\varepsilon > 0$, there exists a partition of $\mathcal{E}$ into such mutually disjoint sets $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_R$ ($R \geq 2$) that $\text{mes } \mathcal{E}_j < \varepsilon$ ($j = 1, 2, \dots, R$). By $L_1^+(G)$ we shall denote the set of nonnegative functions from $L_1(G)$.

Definition 1. A function $M(x, w)$, defined on the set $G \times (-\infty, \infty)$, will be called a **variable N -function** if, for each fixed $x \in G$, it is an N -function of the argument w on the interval $(-\infty, \infty)$ (see ⁽¹⁾, p. 16), and, for each fixed $w \in (-\infty, \infty)$, it is measurable on G as a function of the argument x . For every variable N -function $M(x, w)$, by $M^*(x, w)$ we shall denote the function which, for each fixed $x \in G$, is the complementary function with respect to the N -function $M(x, w)$ (see ⁽¹⁾, p. 22). It is easy to prove that $M^*(x, w)$ is also a variable N -function and that $M^{**}(x, w) = M(x, w)$. By $\mu(x, w)$ we shall denote the function which, for each fixed $x \in G$, is the right derivative of the N -function $M(x, w)$ with respect to the argument w , and by $\mu^*(x, w)$ the function which, for each fixed $x \in G$, is the right derivative of the N -function $M^*(x, w)$ with respect to the argument w .

Definition 2. We shall say that a variable N -function $M(x, w)$ **satisfies on G the Δ'_2 -condition** if the inequality

$$M(x, 2w) \leq f(x) + CM(x, w) \quad (x \in G, -\infty < w < \infty),$$

is fulfilled, where $f(x) \in L_1^+(G)$, and C is some positive number.

Theorem 1. Let $M(x, w)$ be a variable N -function. Then the following three assertions a), b), and c) are equivalent: a) the inequality

$$w\mu(x, w) \leq f_0(x) + C_0M(x, w) \quad (w \geq 0, x \in G),$$

holds, where $f_0(x) \in L_1^+(G)$, C_0 is some positive number; b) the inequality

$$(1 + \gamma)M^*(x, w) \leq \psi_0(x) + w\mu^*(x, w) \quad (w \geq 0, x \in G),$$

holds, where $\psi_0(x) \in L_1^+(G)$, γ is some positive number; c) $M(x, w)$ satisfies on G the Δ'_2 -condition.

Theorem 2. If the variable N -function $M(x, w)$ satisfies on G the Δ'_2 -condition, then the inequality

$$M^*(x, kw) \leq k^{1+\gamma}M^*(x, w) + \psi(x) \quad (w \geq 0, x \in G, 0 \leq k \leq 1),$$

where $\psi(x) \in L_1^+(G)$, γ is some positive number.

For the case when $M(x, w) = M(w)$, $\text{mes } G < \infty$, and $M(w)$ satisfies the Δ_2 -condition, Theorem 2 was formulated in the article of Ya. B. Rutitskii ⁽²⁾.

Definition 3. Let $M(x, w)$ be a variable N -function. By $K_M(G)$ we shall denote the totality of all such real measurable functions $w(x)$, defined on G , for which

$$\rho(w; M) = \int_G M(x, w(x)) dx < \infty.$$

The set $K_M(G)$ will be called an **Orlicz class**.

Theorem 3. In order that the variable N -function $M(x, w)$ satisfy on G the Δ_2 -condition, it is sufficient that at least one of the following inequalities hold:

$$\text{a) } \text{vrai sup}_{x \in G} \left(\sup_{w > f_0(x)} \frac{w\mu(x, w)}{M(x, w)} \right) < \infty, \quad \text{where } f_0(x) \geq 0 \text{ and } f_0(x) \in K_M(G);$$

$$\text{b) } \text{vrai inf}_{x \in G} \left(\inf_{w > \psi_0(x)} \frac{w\mu^*(x, w)}{M^*(x, w)} \right) > 1, \quad \text{where } \psi_0(x) \geq 0 \text{ and } \psi_0(x) \in K_{M^*}(G);$$

$$c) \quad \text{vrai} \inf_{x \in G} \left(\inf_{n > \varphi_0(x)} \frac{w\mu^*(x, w)}{M^*(x, w)} \right) > 1, \quad \text{where } \varphi_0(x) \geq 0 \text{ and } \mu^*(x, \varphi_0(x)) \in K_M(G).$$

Definition 4. Let $M(x, w)$ be a variable N -function. By $L_M(G)$ we shall denote the totality of all such real measurable functions $w(x)$, defined on G , for which

$$\|w\|_{L_M(G)} = \sup_{\rho(v; M^*) \leq 1} \int_G |w(x)| |v(x)| dx < \infty. \quad (1)$$

The set $L_M(G)$ will be called an **Orlicz space**. As was noted in ⁽³⁾, $L_M(G)$, with the norm (1) defined on it, is a Banach space.

Definition 5. We shall say that a sequence of functions $w_n(x) \in L_M(G)$ ($n = 1, 2, \dots$) **converges in mean** to the function $w_0(x) \in L_M(G)$ if

$$\lim_{n \rightarrow \infty} \rho(w_n - w_0; M) = 0.$$

A set $\mathfrak{M} \subset K_M$ will be called **bounded in mean** if

$$\sup_{w(x) \in \mathfrak{M}} \rho(w; M) < \infty.$$

Definition 6. Let $M(x, w)$ be a variable N -function. Since G has σ -finite measure, there exists a sequence $\tilde{G}_k$ ($k = 1, 2, \dots$) of measurable sets of finite measure, nested one in another, such that

$$\bigcup_{k=1}^{\infty} \tilde{G}_k = G.$$

For any pair of natural numbers m and l , evidently, there is a measurable set $G_{ml} \subset \tilde{G}_l$ such that

$$\text{mes}(\tilde{G}_l - G_{ml}) \leq 2^{-(m+l)}$$

and on the set G_{ml} the function $M(x, m)$ is bounded. Put

$$G_k = \bigcup_{l=1}^k \left(\bigcap_{m=1}^{\infty} G_{ml} \right).$$

It is clear that

$$G_1 \subset G_2 \subset \dots \subset G_k \subset \dots, \quad \text{mes } G_k < \infty$$

$$(k = 1, 2, \dots) \quad \text{and} \quad \bigcup_{k=1}^{\infty} G_k = G.$$

We shall regard the sequence of sets G_k ($k = 1, 2, \dots$) as fixed. We shall say that a function $w(x) \in \mathfrak{A}_M(G)$ if its values are only rational numbers, it is measurable, finite-valued, and there exists a natural number k_0 , depending, generally speaking, on $w(x)$, such that $w(x) = 0$ if $x \in G_{k_0}$. It is clear that

$$\mathfrak{A}_M(G) \subset L_M(G).$$

By $E_M(G)$ we shall denote the closure of the set $\mathfrak{A}_M(G)$ in the space $L_M(G)$ in the metric (1).

We now formulate a number of theorems on Orlicz spaces and classes.

Theorem 4. The inclusions $\mathfrak{A}_M(G) \subset E_M(G) \subset K_M(G) \subset L_M(G)$ hold, and the set $\mathfrak{A}_M(G)$ is dense in the Orlicz class $K_M(G)$ in the sense of convergence in the mean.

Theorem 5. For every functional $f \in (E_M(G))^*$ there exists, and moreover is unique, a function $v(x) \in L_{M^*}(G)$ such that the equality

$$f(w(x)) = \int_G v(x)w(x) dx \quad (2)$$

holds for all $w(x) \in E_M(G)$.

Theorem 6. If there exists a not more than countable collection G'_ν ($\nu = 1, 2, \dots$) of measurable subsets of the set G such that, for any set $\mathcal{E} \subset G$ of finite measure,

$$\inf_{\nu} (\text{mes}((\mathcal{E} - G'_\nu) \cup (G'_\nu - \mathcal{E}))) = 0,$$

then the space $E_M(G)$ is separable.

Theorem 7. Let the variable N -function $M(x, w)$ satisfy the inequality

$$M(x, kw) \leq k\delta(k)M(x, w) + \psi(x)$$

($x \in G$, $w > 0$, $0 < k < k_0$), where $\psi(x) \in L_1^+(G)$; $k_0 > 0$; $\delta(k)$ is a nonnegative function defined on the interval $(0, k_0]$ such that $\lim_{k \rightarrow 0} \delta(k) = 0$. Then the relation

$$\lim_{\|w\|_{L_M(G)} \rightarrow \infty} \frac{\rho(w; M)}{\|w\|_{L_M(G)}} = \infty \quad (3)$$

holds.

In the case where $\text{mes } G < \infty$ and $M(x, w) = M(w)$, a necessary and sufficient condition for (3) to hold was obtained by Ya. B. Rutitskii in ⁽²⁾.

Theorem 8. Let the variable N -function $M(x, w)$ satisfy on G the Δ'_2 -condition. Then: a) convergence in norm in the space $L_M(G)$ coincides with convergence in the mean; b) $E_M(G) = K_M(G) = L_M(G)$; c) the general form of a linear functional on $L_M(G)$ is given by equality (2);

d)

$$\lim_{\|w\|_{L_{M^*}(G)} \rightarrow \infty} \frac{\rho(w; M^*)}{\|w\|_{L_{M^*}(G)}} = \infty;$$

e) every set $\mathfrak{M} \subset L_M(G)$ bounded in norm is also bounded in the mean.

Theorem 9. Let G be a space with a nonatomic measure and let the variable N -function $M(x, w)$ not satisfy on G the Δ'_2 -condition. Then: a) the Orlicz class $K_M(G)$ is not a linear set (and consequently $E_M(G) \neq K_M(G)$, $K_M(G) \neq L_M(G)$, and $E_M(G) \neq L_M(G)$); b) there exists a functional $f_0 \in (L_M(G))^*$ for which there is no function $v(x)$ such that for all $w(x)$ from $L_M(G)$, $v(x)w(x) \in L_1(G)$ and the equality

$$f_0(w(x)) = \int_G v(x)w(x) dx$$

holds; in other words, (2) is not the general form of a linear functional on $L_M(G)$.

Theorem 10. In order that the Orlicz space $L_M(G)$ be reflexive, it is sufficient, and if G is a space with a nonatomic measure, also necessary, that each of the variable N -functions $M(x, w)$ and $M^*(x, w)$ satisfy on G the Δ'_2 -condition.

Theorem 11. Let $M_1(x, w)$ and $M_2(x, w)$ be variable N -functions satisfying on G the Δ'_2 -condition; let $\varphi(x, w)$ be a function satisfying the Carathéodory condition, and suppose the inequality

$$M_2(x, \varphi(x, w)) \leq f(x) + CM_1(x, w) \quad (x \in G, -\infty < w < \infty), \quad (4)$$

holds, where $f(x) \in L_1^+(G)$, and C is some positive number. Then the operator Φ , defined by the equality $\Phi w(x) = \varphi(x, w(x))$, maps $L_{M_1}(G)$ into $L_{M_2}(G)$ and is continuous.

Theorem 12. Let G be a space with a nonatomic measure; $M_1(x, w)$ and $M_2(x, w)$ be variable N -functions satisfying on G the Δ'_2 -condition; let $\varphi(x, w)$

be a function satisfying the Carathéodory condition, and suppose $\varphi(x, w(x)) \in K_{M_2}(G)$ for every function $w(x) \in K_{M_1}(G)$. Then the opera-

the operator Φ , defined by the equality $\Phi w(x) = \varphi(x, w(x))$, maps $L_{M_1}(G)$ into $L_{M_2}(G)$, is continuous, and is bounded on every ball of the space $L_{M_1}(G)$, and for the function $\varphi(x, w)$ the inequality

$$M_2(x, \varphi(x, w)) \leq f(x) + CM_1(x, w) \quad (x \in G, -\infty < w < \infty),$$

holds, where $f(x) \in L_1^+(G)$, and C is some positive number.

Remark. Theorems 9 and 12 are proved on the basis of a result of M. A. Krasnosel'skii ((⁷), p. 26).

The following theorem is a generalization of the well-known Vitali theorem (see (⁸), p. 167).

Theorem 13. Let the variable N -function $M(x, w)$ satisfy the Δ'_2 -condition on G . Let, further, $w_n(x)$ ($n = 1, 2, \dots$) be a sequence of functions from the space $L_M(G)$, and let $w_0(x)$ be some real-valued measurable and almost everywhere finite function on G . In order that $w_0(x) \in L_M(G)$ and

$$\lim_{n \rightarrow \infty} \|w_n(x) - w_0(x)\|_{L_M(G)} = 0,$$

it is necessary and sufficient that the following conditions be satisfied: a) the sequence $w_n(x)$ ($n = 1, 2, \dots$) converges to the function $w_0(x)$ in measure on every set of finite measure $\mathcal{E} \subset G$; b) the equalities

$$\inf_{\{\mathcal{E}: \text{mes } \mathcal{E} < \infty\}} \left(\sup_{n \geq 1} \int_{G \setminus \mathcal{E}} M(x, w_n(x)) dx \right) = 0,$$

$$\inf_{\delta > 0} \left(\sup_{\{\mathcal{E}: \text{mes } \mathcal{E} < \delta\}} \left(\sup_{n \geq 1} \int_{\mathcal{E}} M(x, w_n(x)) dx \right) \right) = 0.$$

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