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Abstract

Full Text

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MATHEMATICS

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ON ADDITIVE FUNCTIONS OF SETS CONTINUOUS IN A DIRECTION

(Presented by Academician S. L. Sobolev on 10 III 1966)

We shall consider countably additive real functions of sets $\Phi(E)$, defined on Borel sets E of the n -dimensional cube Ω with sides parallel to the coordinate axes $x_1, x_2, \dots, x_n$. For an arbitrary E from n -dimensional Euclidean space R_n we shall put $\Phi(E) = \Phi(E \cap \Omega)$. By $W\Phi(E)$ we shall denote the total variation of the function Φ on E . Finally, together with the function $\Phi(E)$, we shall consider the function $\Phi^h(E) = \Phi(E + h)$, where h is a vector in R_n , and $E + h$ denotes the set E shifted as a rigid body by the vector h .

Classical results (see, for example, ⁽¹⁾) assert the equivalence of the following three conditions:

1°. The function $\Phi(E)$ is absolutely continuous with respect to Lebesgue measure.

2°.

$$\lim_{|h| \rightarrow 0} W[\Phi^h - \Phi](\Omega) = 0 \quad (1)$$

(the function $\Phi(E)$ is continuous with respect to translation).

3°.

$$\Phi(E) = \int_E f(x) dx, \quad x \in R_n. \quad (2)$$

The present note is devoted to countably additive functions possessing properties 1° or 2° only along certain coordinate axes. In this case, as we shall show, conditions 1° and 2° cease to be equivalent. Functions possessing these properties are, generally speaking, singular. However, for them one can obtain certain integral representations analogous to (2).

Let $m_k A$ denote the k -dimensional Lebesgue measure of the set A , and let $P_i A$ be the projection of the set A onto the axis x_i .

Definition 1. The function $\Phi(E)$ is called **absolutely continuous in the coordinate direction x_i** if $|\Phi(E)| \rightarrow 0$ as soon as $m_1 P_i E \rightarrow 0$.

Example 1. The function of plane sets

$$\varphi(E) = m_1 P_1(E \cap d) = m_1 P_2(E \cap d),$$

where d is the diagonal of the unit square, is an example of a function absolutely continuous in each coordinate direction.

The function $\varphi(E)$ is obviously singular. Thus, absolute continuity in each coordinate direction is insufficient for ordinary absolute continuity.

Let $P_{R_s} E$ denote the projection of the set E onto the subspace R_s , $s < n$.

Definition 2. The function $\Phi(E)$ is called **absolutely continuous with respect to the collection of coordinate directions** $x_1, x_2, \dots, x_s$, $s < n$, if $|\Phi(E)| \rightarrow 0$ as soon as $m_s P_{R_s} E \rightarrow 0$.

From absolute continuity with respect to the collection $x_1, x_2, \dots, x_s$ it obviously follows that there is absolute continuity in each of the directions $x_1, x_2, \dots, x_s$. The converse is not true.

Let $R_s^\perp$ denote the orthogonal complement of R_s , $s < n$, in R_n , and let BG be the totality of all Borel sets from some set $G \subset R_n$.

Theorem 1. In order that the function $\Phi(E)$ be absolutely continuous with respect to the totality of directions $x_1, x_2, \dots, x_s$, $s < n$, it is necessary and sufficient that one of the following conditions be fulfilled:

- 1) $\Phi(E) = 0$, if $m_s P_{R_s} E = 0$.
- 2) For sets E of the form $E = E_s \times E_s^\perp$, where $E_s \in BR_s$, and $E_s^\perp \in BR_s^\perp$, the representation holds

$$\Phi(E_s \times E_s^\perp) = \int_{E_s} f(E_s^\perp, x^s) dx^s,$$

where $x^s \in R_s$; $f(E_s^\perp, x^s)$, for almost all x^s , is countably additive in $E_s^\perp$ and, for all $E_s^\perp \in BR_s^\perp$, is summable in x^s .

It follows from 1) that from the set of singularities of a function $\Phi(E)$ absolutely continuous with respect to the totality of directions $x_1, x_2, \dots, x_s$, one may remove any set Q such that $m_s P_{R_s} Q = 0$, without this set ceasing to be a set of singularities.

Theorem 2. If $\Phi(E)$ is absolutely continuous with respect to the totality of directions $x_1, x_2, \dots, x_s$, then the functions $\overline{W}\Phi(E)$, $\underline{W}\Phi(E)$, and $W\Phi(E)$ will have the same property. For $W\Phi(E)$ the representation holds

$$W\Phi(E_s \times E_s^\perp) = \int_{E_s} Wf(E_s^\perp, x^s) dx^s.$$

Analogous representations hold for the upper and lower variations $\overline{W}\Phi(E)$ and $\underline{W}\Phi(E)$.

Let $x_s^\perp \in R_s^\perp$. Denote $\{x_s^\perp\} = x_s^\perp \times R_s$.

Definition 3. The function $\Phi(E)$ is called **continuous under shifts with respect to the totality of directions** $x_1, x_2, \dots, x_s$, if $|\Phi(E)| \rightarrow 0$ as soon as $m_s[\{x_s^\perp\} \cap E] \rightarrow 0$ simultaneously for all $\{x_s^\perp\}$.

Lemma. Let Q be a set from $B\Omega$ such that for any $\{x_s^\perp\}$ the measure $m_s[Q \cap \{x_s^\perp\}] = 0$. Then for almost all, in the sense of the measure m_s , vectors $h \in R_s$, the equality

$$\Phi^h(Q) = \Phi(Q + h) = 0$$

is valid.

Theorem 3. In order that the function $\Phi(E)$ be continuous under shifts with respect to the totality of directions $x_1, x_2, \dots, x_s$, $s < n$, it is necessary and sufficient that one of the following conditions be fulfilled:

1. The function $\Phi(E)$ is absolutely continuous with respect to the measure $\psi(E)$, defined by the equality

$$\psi(E_s \times E_s^\perp) = W\Phi(\Omega_s \times E_s^\perp) \cdot m_s E_s,$$

where $\Omega_s = P_{R_s} \Omega$, and, consequently,

$$\Phi(E) = \int_E f(x) d\psi(E).$$

2. The function $\Phi(E)$ satisfies condition (1) only for vectors $h \in R_s$.
3. The function $\Phi(E)$ vanishes on any set $E \in BR_n$ such that $m_s[E \cap \{x_s^\perp\}] = 0$ for any $\{x_s^\perp\}$.

Thus, any of the properties 1, 2, and 3 listed in Theorem 3 may be taken as the definition of a function continuous under shifts.

Corollary 1. If $\Phi(E)$ is continuous under shifts with respect to the totality $x_1, x_2, \dots, x_s$, then it is absolutely continuous with respect to this totality.

The converse, as we have seen, is false.

Corollary 2. The projection onto $R_s^\perp$ of the set of singularities of a function $\Phi(E)$ that is continuous under shifts jointly in $x_1, x_2, \dots, x_s$ has $(n - s)$ -dimensional measure zero.

Corollary 3. If the function $\Phi(E)$ is continuous under shifts jointly in $x_1, x_2, \dots, x_s$ and absolutely continuous jointly in $x_{s+1}, x_{s+2}, \dots, x_n$, then it is absolutely continuous in the usual sense.

Example 2. The function $\eta(E)$, defined for sets $E = E_s \times E_s^\perp$ by the equality

$$\eta(E) = \gamma(E_s) \cdot \delta(E_s^\perp),$$

is continuous under shifts jointly in $x_1, x_2, \dots, x_s$, if $\gamma(E_s)$ is an absolutely continuous set function, $E_s \in BR_s$, and $\delta(E_s^\perp)$ is a countably additive set function, $E_s^\perp \in BR_s^\perp$.

The following example shows that the simultaneous fulfillment of Corollaries 1 and 2 is only a necessary condition for shift continuity of the function $\Phi(E)$.

Example 3. In the plane square $\Omega = \{0 \leq x \leq 1, 0 \leq y \leq 1\}$ construct the graph of the “Cantor staircase” $y = f(x)$ and remove from this graph the intervals on which the function $f(x)$ is constant. Denote by C the remaining set of points on the graph, and define a set function $E \subset \Omega$ by the formula

$$\varphi(E) = m_1 P_y(E \cap C),$$

where P_y denotes projection onto the y -axis. The function thus constructed is, obviously, absolutely continuous in the y -direction, since $\varphi(E) = 0$ if $m_1 P_{yE} = 0$. The projection of its set of singularities (the set C) onto the x -axis has measure zero. At the same time, $\varphi(E)$ is not continuous under shifts along y .

The properties of set functions considered here are connected with differentiation in the generalized sense of S. L. Sobolev (see (2), and also (3)). We indicate two simple propositions:

1. If a countably additive set function $\Phi(E)$ has, as generalized derivatives in the directions $x_1, x_2, \dots, x_s$, also countably additive functions, then $\Phi(E)$ is continuous under shifts jointly in $x_1, x_2, \dots, x_s$.
2. If a countably additive function $\Phi(E)$ is absolutely continuous jointly in $x_2, x_3, \dots, x_n$, then as its primitive in the direction x_1 one may take a set function that is absolutely continuous in the usual sense.

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