

ON INVERSION
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FUNDAMENTAL
INTEGRAL
REPRESENTATION OF
 $\backslash(p\backslash)$ -ANALYTIC
FUNCTIONS OF
 $\backslash(z=x+iy\backslash)$ WITH
CHARACTERISTIC
 $\backslash(p=x^k\backslash)$

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1967

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Abstract

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UDC 517.53:512.9

MATHEMATICS

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ON INVERSION FORMULAS FOR THE FUNDAMENTAL INTEGRAL REPRESENTATION OF p -ANALYTIC FUNCTIONS OF $z = x + iy$ WITH CHARACTERISTIC $p = x^k$

(Presented by Academician I. N. Vekua, 12 I 1967)

Let G be a domain in the right half-plane $z = x + iy$, and let a be a point on its boundary S . The fundamental integral representation of x^k -analytic functions of $z = x + iy$ ($x > 0$, $k = \text{const} > 0$) has the form ⁽¹⁻³⁾

$$\begin{aligned} \tilde{f}(z) &= \tilde{u}(x, y) + i\tilde{v}(x, y) \\ &= \text{Re} \int_a^z f(\zeta) \left(\frac{z + \bar{z}}{2} \right)^{1-k} (z - \zeta)^{k/2-1} (\bar{z} + \zeta)^{k/2-1} d\zeta \\ &\quad + i \text{Im} \int_a^z f(\zeta) \left(\zeta - \frac{z - \bar{z}}{2} \right) (z - \zeta)^{k/2-1} (\bar{z} + \zeta)^{k/2-1} d\zeta, \end{aligned} \quad (1)$$

where the integration from a to z is carried out along any contour Γ lying in G ; $\arg(z - \zeta)(\bar{z} + \zeta)$ is chosen in one way or another; $f(z) = u(x, y) + iv(x, y)$ and $\tilde{f}(z) = \tilde{u}(x, y) + i\tilde{v}(x, y)$ are analytic and, respectively, x^k -analytic functions in the domain G . In this case three cases are essentially distinguished: a) S contains a segment L of the imaginary axis and $v|_L = 0$, while a is an arbitrary (unfixed) point on L ; b) $a = \infty$ and, as $z \rightarrow \infty$, $f(z) = O(|z|^{-k-\varepsilon})$ ($\varepsilon = \text{const} > 0$); c) $a \neq \infty$ and, as $z \rightarrow a$, $f(z) = O(|z - a|^{-1+\varepsilon})$.

Introducing the "adjoint" p -analytic function with characteristic $p = x^k$,

$$f^*(z) = u^*(x, y) + iv^*(x, y) = \tilde{u}(x, y) + \frac{1}{x^k} \tilde{v}(x, y),$$

equality (1) can be written in the equivalent form

$$\begin{aligned} f^*(z) &= u^*(x, y) + iv^*(x, y) \\ &= \frac{1}{2x^k} \int_a^z f(\zeta)(z - \zeta)^{k/2-1}(\bar{z} + \zeta)^{k/2} d\zeta + \overline{f(\zeta)}(\bar{z} - \bar{\zeta})^{k/2}(z + \bar{\zeta})^{k/2-1} d\bar{\zeta}. \end{aligned} \quad (2)$$

From the conditions of p -analyticity it follows that ^(4,5)

$$\frac{\partial}{\partial \bar{z}} f^* = -\frac{k}{2(z + \bar{z})} (f^* - \bar{f}^*), \quad (3)$$

i.e. $f^*(z)$ is a generalized analytic function ⁽⁶⁾ or a generalized analytic function of the fourth class ⁽⁵⁾.

In cases b) and c) we introduce some additional assumptions. Instead of case b) we shall speak of case b'), assuming that in G there exists a rectilinear horizontal segment C issuing from the point $a = \infty$, and that on this segment, as $z \rightarrow \infty$, $f(z)$, $df(z)/dz = O(x^{-2-\varepsilon})$. From case c) we single out two cases c') and c''), assuming that in G there exists a rectilinear horizontal segment C issuing from the point a to the right or, respectively, to the left, and that $f(z)$, $df(z)/dz = O(1)$ as $z \rightarrow a$.

It is easy to establish that

$$\begin{aligned} \tilde{F}(z) &= \tilde{U}(x, y) + i\tilde{V}(x, y) = \\ &= x^{k-1}\tilde{u}(x, y) + i \int_a^z x^{1-k} \frac{\partial \tilde{v}(x, y)}{\partial x} dx + \left(x^{1-k} \frac{\partial \tilde{v}(x, y)}{\partial y} + (k-1)\tilde{u}(x, y) \right) dy \end{aligned} \quad (4)$$

will be an $x^{k'}$ -analytic function in G , where $k' = 2 - k$.

If $0 < k < 2$ and

$$F^*(z) = U^*(x, y) + iV^*(x, y) = \widehat{U}(x, y) + i\frac{1}{x^k}V(x, y)$$

is the adjoint $x^{k'}$ -analytic function, then the integral

$$I(z) = \int_a^z F^*(t)(t - z)^{k'/2-1}(\bar{t} + z)^{k'/2} dt + \overline{F^*(t)}(t - z)^{k'/2}(\bar{t} + z)^{k'/2-1} d\bar{t} \quad (5)$$

does not depend on the contour of integration Γ connecting the points a and z , and is a function of z analytic in G .

The first part of the assertion follows from the identity (5)

$$\frac{\partial}{\partial t} \left(F^*(t)(t-z)^{k'/2-1}(\bar{t}+z)^{k'/2} \right) = \frac{\partial}{\partial \bar{t}} \left(\overline{F^*(t)}(t-z)^{k'/2}(\bar{t}+z)^{k'/2-1} \right),$$

the second part of the assertion—from the equality

$$I(z) \left(1 - e^{\mp i 2\pi(k'/2-1)} \right) = \int_{\gamma} F^*(t)(t-z)^{k'/2-1}(\bar{t}+z)^{k'/2} dt + \\ + \overline{F^*(t)}(t-z)^{k'/2}(\bar{t}+z)^{k'/2-1} d\bar{t},$$

where γ is the contour going from the point a along the left (or right) edge of the cut along Γ , encircling the point z in the negative (or positive) direction and returning to the point a along the right (or left) edge of the cut.

A. Consider cases a) and b'). Taking $\arg(z-\zeta)(\bar{z}+\zeta) = 0$ on the upper (left) edge of the cut along C and understanding by the values of the functions their values on the indicated edge of the cut along C or Γ , introduce the function analytic in G

$$f_1(z) = u_1(x, y) + iv_1(x, y) = \mu \frac{1}{2e^{i\pi(k'/2-1)}} \frac{d}{dz} I(z), \quad (6)$$

where $\mu = 2\Gamma^{-1}(-k/2+1)\Gamma^{-1}(k/2)$.

On the segment C , equalities (6) and (1) take the form

$$f_1(z) = u_1(x, y) + iv_1(x, y) = \\ = \mu \frac{d}{dx} \int_{x_0}^x \left[\tilde{u}(\xi, y) \xi^k + i \frac{x}{\xi^{2-k}} V(\xi, y) \right] \frac{d\xi}{(x^2 - \xi^2)^{k/2}}, \quad (7)$$

$$\tilde{f}(z) = \tilde{u}(x, y) + i\tilde{v}(x, y) = \int_{x_0}^x [x^{1-k} u(\xi, y) + i\xi v(\xi, y)] (x^2 - \xi^2)^{k/2-1} d\xi, \quad (8)$$

where $x_0 = \operatorname{Re} a$.

From (8) and (4) it follows that

$$\tilde{v}(x, y)|_{z=a} = 0, \quad \tilde{V}(x, y)|_{z=a} = 0. \quad (9)$$

The inversion formula (8) has the form ^(5,7)

$$u(x, y) + ixv(x, y) = \mu \frac{d}{dx} \int_{x_0}^x [i\tilde{u}(\xi, y)\xi^{k-1} + \tilde{v}(\xi, y)] \frac{\xi d\xi}{(x^2 - \xi^2)^{k/2}}. \quad (10)$$

Comparing (7), (10) and taking (9) into account, we find on C

$$f_1(z) = f(z) = \mu \frac{d}{dx} \int_{x_0}^x \tilde{u}(\xi, y) \frac{\xi^k d\xi}{(x^2 - \xi^2)^{k/2}} + i\mu x^{1-k} \int_{x_0/x}^1 \frac{\partial \tilde{v}(x\beta, y)}{\partial x\beta} \frac{d\beta}{(1 - \beta^2)^{k/2}}.$$

Consequently, $f_1(z) \equiv f(z)$ in the domain G , and the desired inversion formula for the basic integral representation (1) is written in the form

$$f(z) = u(x, y) + iv(x, y) = \quad (11)$$

$$= \mu \frac{1}{2e^{i\pi(k'/2-1)}} \frac{d}{dz} \int_a^z F^*(t)(t-z)^{k'/2-1}(\bar{t}+z)^{k'/2} dt + \bar{F}^*(t)(t-z)^{k'/2}(\bar{t}+z)^{k'/2-1} d\bar{t}$$

or

$$f(z) = u(x, y) + iv(x, y) = \mu \frac{1}{2e^{i\pi(k'/2-1)}} \frac{d}{dz} \int_a^z \tilde{U}(\xi, \eta) d\tilde{Z} + i\tilde{V}(\xi, \eta) dZ, \quad (12)$$

where Z and $\tilde{Z}$ are the conjugate variables ^(4,5) of $t = \xi + i\eta$ with characteristic $p = \xi^{k'}$,

$$\tilde{Z} = \frac{2}{k'}(t-z)^{k'/2}(\bar{t}+z)^{k'/2}, \quad dZ = 2^{k'+1} \left(\sqrt{\frac{t-z}{t+z}} + \sqrt{\left(\frac{\bar{t}+z}{t-z}\right)^{-k'}} \right) d \ln \sqrt{\frac{t-z}{\bar{t}+z}}. \quad (12')$$

B. Let us consider cases ') and '). Setting $\arg(z - \xi)(\bar{z} + \xi) = -\pi$ on the upper (right-hand) edge of the cut along C and understanding by the values of the functions their values on the indicated edge of the cut along C or Γ , introduce the function analytic in G

$$f_2(z) = u_2(x, y) + iv_2(x, y) = \mu \frac{1}{2} \frac{d}{dz} I(z). \quad (13)$$

From (13) and (1) we obtain on C

$$f_2(z) = u_2(x, y) + iv_2(x, y) = \mu \frac{d}{dx} \int_{x_0}^x \left[\tilde{u}(\xi, y) \xi^k + i \frac{x}{\xi^{2-k}} \tilde{V}(\xi, y) \right] \frac{d\xi}{(\xi^2 - x^2)^{k/2}}, \quad (14)$$

$$\tilde{f}(z) = \tilde{u}(x, y) + i\tilde{v}(x, y) = \int_{x_0/x}^1 [u_3(x\beta, y) + ix^k \beta v_3(x\beta, y)] (\beta^2 - 1)^{k/2-1} d\beta, \quad (15)$$

where $f_3(z) = u_3(x, y) + iv_3(x, y) = e^{-i\pi(k/2-1)} f(z)$. From (15) it follows that in case ') , when approaching infinity,

$$\tilde{u}(x, y), \quad \partial \tilde{u}(x, y) / \partial x, \quad \partial \tilde{u}(x, y) / \partial y = O(x^{-2-\varepsilon}), \quad \tilde{v}(x, y) = O(x^{k-2-\varepsilon}).$$

Therefore in case ') , as in case ") , the equalities (9) hold. Taking this and the inversion formula for the basic integral representation (1) on $C^{(5, 7)}$ into account,

$$u_3(x, y) + ixv_3(x, y) = \mu \frac{d}{dx} \int_{x_0}^x [\tilde{u}(\xi, y) \xi^{k-1} + i\tilde{v}(\xi, y)] \frac{\xi d\xi}{(\xi^2 - x^2)^{k/2}},$$

we come to the conclusion that on C

$$f_2(z) = f_3(z) = \mu \frac{d}{dx} \int_{x_0}^x \tilde{u}(\xi, y) \frac{\xi^k d\xi}{(\xi^2 - x^2)^{k/2}} + i\mu x^{1-k} \int_{x_0/x}^1 \frac{\partial \tilde{v}(x\beta, y)}{\partial x\beta} \frac{d\beta}{(\beta^2 - 1)^{k/2}}.$$

Thus, the desired inversion formula for the basic integral representation (1) is obtained in the following form:

$$\begin{aligned} f(z) &= u(x, y) + iv(x, y) = \\ &= \mu \frac{1}{2e^{i\pi k'/2}} \frac{d}{dz} \int_a^z F^*(t) (t-z)^{k'/2-1} (\bar{t}+z)^{k'/2} dt \\ &\quad + \bar{F}^*(t) (t-z)^{k'/2} (\bar{t}+z)^{k'/2-1} d\bar{t} \end{aligned} \quad (16)$$

or

$$f(z) = u(x, y) + iv(x, y) = \mu \frac{1}{2e^{i\pi k'/2}} \frac{d}{dz} \int_a^z \tilde{U}(\xi, \eta) d\tilde{Z} + i\tilde{V}(\xi, \eta) dZ, \quad (17)$$

where Z and $\tilde{Z}$ are conjugate variables defined by the equalities (12'), under the agreement made above on the choice of $\arg(z - \zeta)(\bar{z} + \zeta)$.

C). Let now $k = 2m + k_1$, where m is a positive integer, $0 < k_1 < 2$. From (2) it follows that the function

$$f_1^*(z) = u_1^*(x, y) + iv_1^*(x, y) = \frac{1}{\alpha x^{k_1}} \frac{d^{2m}}{dz^m d\bar{z}^m} (x^k f^*(z)),$$

$$\alpha = \left(\frac{k}{2} - 1\right) \frac{k}{2} \left(\frac{k}{2} - 2\right) \left(\frac{k}{2} - 1\right) \dots \frac{1}{2} \frac{3}{2} \quad (18)$$

will be adjoint x^{k_1} -analytic in G . Introducing the x^{k_1} -analytic function

$$f_1(z) = \tilde{u}_1(x, y) + i\tilde{v}_1(x, y) = u_1^*(x, y) + ix^{k_1}v_1^*(x, y),$$

we arrive at the conclusion that the inversion formula of the principal integral representation (1) for $k = 2m + k_1$ is given by equality (11) or (12) in cases a) and b'), and by equality (16) or (17) in cases b) and c'), provided that in these equalities k is replaced by k_1 , and $\tilde{u}(x, y)$ and $\tilde{v}(x, y)$ are replaced by $\tilde{u}_1(x, y)$ and $\tilde{v}_1(x, y)$.

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Received
4 I 1967

CITED LITERATURE

- ¹ G. M. Polozhii, *Scientific Yearbook of Kyiv University for 1957*, Kyiv, 1958, p. 340.
- ² G. M. Polozhii, *Bulletin of Kyiv University*, No. 2, series astr., matem., mekh., issue 1, 1959, p. 19.
- ³ G. N. Polozhii, *Ukrainian Mathematical Journal*, 16, issue 2 (1964).
- ⁴ G. N. Polozhii, *DAN*, 58, No. 7 (1947).
- ⁵ G. N. Polozhii, *Generalization of the Theory of Analytic Functions of a Complex Variable. p -Analytic and (p, q) -Analytic Functions and Some of Their Applications*, Kyiv, 1965.
- ⁶ I. N. Vekua, *Generalized Analytic Functions*, Moscow, 1959.
- ⁷ G. N. Polozhii, *Ukrainian Mathematical Journal*, 16, No. 5 (1964).

Note: Figure translations are in progress. See original paper for figures.

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