

THE CAUCHY PROBLEM AND A BOUNDARY-VALUE PROBLEM FOR LINEAR AND QUASILINEAR DEGENERATE HYPERBOLIC EQUATIONS OF SECOND ORDER

G. M. Fateeva

1967

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196701.45777>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.946

MATHEMATICS

G. M. FATEEVA

THE CAUCHY PROBLEM AND A BOUNDARY-VALUE PROBLEM FOR LINEAR AND QUASILINEAR DEGENERATE HYPERBOLIC EQUATIONS OF SECOND ORDER

(Presented by Academician I. G. Petrovskii, 29 IV 1966)

The Cauchy problem and the boundary-value problem for linear hyperbolic equations of second order degenerating in the domain and on the boundary were studied in the work of O. A. Oleinik ⁽¹⁾. In the case of an equation degenerating only on the boundary of the domain, these problems have been considered by many authors (see the literature cited in ⁽¹⁾).

I. The Cauchy problem. In the domain $G_T = \{0 \leq t \leq T, x \in R_n(x_1, \dots, x_n)\}$ we consider the Cauchy problem for the equation

$$Lu \equiv -u_{tt} + a^{ij}(t, x)u_{x_i x_j} + b^i(t, x)u_{x_i} + b^0(t, x)u_t + c(t, x)u = f(t, x), \quad (1)$$

where $a^{ij} = a^{ji}$, $a^{ij}(t, x)\xi_i \xi_j \geq 0$ *, for $(t, x) \in G_T$ and all $\xi = (\xi_1, \dots, \xi_n)$, with zero initial conditions

$$u(0, x) = 0, \quad u_t(0, x) = 0. \quad (2)$$

For simplicity it is assumed that $f(t, x)$ is a finite function with respect to x .

Applying the methods of the works ^(1, 2), as well as the methods developed by S. L. Sobolev ⁽³⁾, we shall indicate smoothness conditions on the coefficients of equation (1) which ensure the existence of a classical solution of the problem (1), (2), which also permits consideration of quasilinear equations of the form

$$-u_{tt} + a^{ij}(t, x, u)u_{x_i x_j} + b^i(t, x, u)u_{x_i} + b^0(t, x, u)u_t = f(t, x, u), \quad (3)$$

$$a^{ij}(t, x, u)\xi_i \xi_j \geq 0 \quad \text{for } (t, x) \in G_T, \quad |u| \leq M, \quad M = \text{const} > 0.$$

Let

$$D = \partial^k / \partial t^{k_0} \partial x_1^{k_1} \dots \partial x_n^{k_n} \quad (k_0 + \dots + k_n = k), \quad D_x^k = \partial^k / \partial x_1^{k_1} \dots \partial x_n^{k_n} \quad (k_1 + \dots + k_n = k).$$

Theorem 1. In order that the Cauchy problem for equation (1) with conditions (2) have a solution continuous together with derivatives up to order $m \geq 2$, it is sufficient that the following conditions be satisfied for $l \geq m + 1 + [(n + 1)/2]$:

- 1) The coefficients a^{ij} and their derivatives $a_{x_k}^{ij}$, a_t^{ij} , $a_{x_k x_r}^{ij}$ are bounded:

$$\max\{|a^{ij}|, |a_t^{ij}|, |a_{x_k}^{ij}|, |a_{x_k x_r}^{ij}|\} \leq A,$$

the derivatives of a^{ij} of orders $2, 3, \dots, l$ satisfy the inequalities

$$\max\{\|D^\beta a^{ij}\|_{L_{p_1}(G)}, \|D_x^\beta a^{ij}\|_{L_{p_2}(G)}\} \leq A_{G_T} \leq A,$$

where $p_s = (n + 1 + \sigma)/(\beta - s)$ for $\beta \leq s + (n + 1)/2$, $p_s = 2$ for $\beta > s + (n + 1)/2$; $s = 1, 2$; $\sigma > 0$.

* Here, as everywhere below, summation over repeated indices from 1 to n is assumed.

- 2) The coefficients $b^i(t, x)$ ($i = 0, \dots, n$) and their derivatives $b_{x_k}^i$, b_t^0 are bounded:

$$\max\{|b^i|, |b_t^0|, |b_{x_k}^i|\} \leq A;$$

moreover, the derivatives of $b^i(t, x)$ up to order l satisfy the inequalities

$$\left\{ \|D^\beta b^i\|_{L_{q_0}(G_T)}, \|D_x^\beta b^i\|_{L_{q_1}(G_T)} \right\} \leq A_{G_T} \leq A,$$

where

$$q_s = (n + 1 + \sigma)/(\beta - s) \quad \text{for } \beta \leq s + (n + 1)/2, \quad q_s = 2 \quad \text{for } \beta > s + (n + 1)/2;$$

$$s = 0, 1; \quad \sigma > 0.$$

- 3) The coefficient $c(t, x)$ is bounded:

$$\max |c(t, x)| \leq A,$$

and the derivatives of $c(t, x)$ up to order l satisfy the inequalities

$$\left\{ \|D^\beta c\|_{L_{r_1}(G_T)}, \|D^\beta c\|_{L_{r_0}(G_T)} \right\} \leq A_{G_T} \leq A,$$

where

$$r_s = (n + 1 + \sigma)/(\beta + s) \quad \text{for } \beta \leq (n + 1)/2 - s, \quad r_s = 2 \quad \text{for } \beta > (n + 1)/2 - s;$$

$$s = 0, 1; \quad \sigma > 0.$$

4) The function $f(t, x)$ and its derivatives up to order l satisfy the inequalities

$$\|D^\beta f(t, x)\|_{L_2(G_T)} \leq F_{G_T} \leq F. \quad D^i f(0, x) = 0, \quad 0 \leq i \leq l-2.$$

In addition, it is assumed that there exist $K > 0$ and $\delta > 0$ such that

$$K a^{ij} \xi_i \xi_j + a_t^{ij} \xi_i \xi_j - \delta [(b^i - a_{x_j}^{ij}) \xi_i]^2 \geq 0 \quad \text{for all } \xi. \quad (4)$$

Remark. Inequality (4) is essential for the correctness of the problem (1), (2) (see (4, 5)).

The proof of Theorem 1 is carried out according to the same scheme as in (1). The main point of the proof is obtaining inequalities of the form

$$U_k(t) \leq C \int_0^t U_k^{1/2}(\tau) \left\{ A \sum_{i=0}^k U_i^{1/2}(\tau) + F \right\} d\tau, \quad (5)$$

where

$$U_k(\tau) = \iint_{G_\tau} \sum_{k_0 + \dots + k_n = k} \left(\frac{\partial^k u}{\partial t^{k_0} \dots \partial x_n^{k_n}} \right)^2 d\tau dx, \quad C = \text{const.}$$

In proving these inequalities one repeatedly uses the embedding theorems of S. L. Sobolev, similarly to (3) (see § 21).

Consider the Cauchy problem for the quasilinear equation (3) with conditions (2). The function $f(t, x, u)$ is assumed to be finite with respect to x , and the support of $f(t, x, u)$ for $t \leq T$, $|u| \leq M$ belongs to the cylinder

$$Q^0 = \{[0, T] \times \Omega_0\}.$$

Let Ω_1 be a bounded domain such that $\Omega_1 \supset \Omega_0$, and let the distance between the boundaries of Ω_1 and Ω_0 be not less than

$$\rho_0 = 1 + T\sqrt{n(A_0 + 1)},$$

where

$$A_0 = \sup |a^{ij}(t, x, u)|$$

for $(t, x) \in G_T$, $|u| \leq M$. Let

$$Q_\tau = \{0 \leq t \leq \tau, x \in \Omega_1\}, \quad 0 < \tau \leq T.$$

Let the function $\Phi(t, x_1, \dots, x_n, u)$ be defined in the $(n+2)$ -dimensional domain

$$E \equiv \{0 \leq t \leq T, x \in \Omega_1, |u| \leq M\},$$

be continuous in E , and have l continuous derivatives with respect to u , while the functions

$$\Phi_{u^k} \equiv \frac{\partial^k}{\partial u^k} \Phi \quad (0 \leq k \leq l)$$

have generalized derivatives up to order l with respect to $t, x_1, \dots, x_n$ for each fixed value of u . Denote by E_u the set of functions continuous in $\bar{Q}_\tau$:

$$u = v(t, x), \quad |v(t, x)| \leq M.$$

Definition. We shall say that the function $\Phi(t, x, u)$ has property $\tilde{T}$ if there exists a number $p > 1$, $p > (n+1)/l$, such that the result of substituting into the function

$$D^\alpha \Phi_{u^k} \equiv \frac{\partial^\alpha}{\partial t^{\alpha_0} \dots \partial x_n^{\alpha_n}} \Phi_{u^k}(t, x, u)$$

instead of u any function from E_u is a composite function of (t, x) belonging, for

$$l \geq \alpha > l - (n+1)/p,$$

to the space

$$L_{\frac{1}{1/p - (l-\alpha)/(n+1)}}(Q_\tau),$$

and moreover

$$\left\| D^\alpha \Phi_{u^k}(t, x, u) \Big|_{u=v(t,x)} \right\|_{L_{\frac{1}{1/p - (l-\alpha)/(n+1)}}} \leq A_{Q_\tau},$$

where the constant A_{Q_τ} does not ...

depends on $v(t, x) \in E_u$, $0 \leq \tau \leq T$. If $\alpha < l < (n+1)/p$, then the space

$$L_{\frac{1}{1/p - (l-\alpha)/(n+1)}}$$

should be replaced by $C(Q_\tau)$, and for $\alpha = l - (n+1)/p$, respectively, by L_q , where $q > 1$ is arbitrary (the constant A_{Q_τ} depends on q).

Lemma 1. If the function $\Phi(t, x, u)$ has property $\tilde{T}$, and the function $u = v(t, x)$ satisfies the condition

$$\iiint_{Q_\tau} |D^\alpha v(t, x)|^{\frac{1}{1/p - (l-\alpha)/(n+1)}} dt dx \leq B_{Q_\tau}^{\frac{1}{1/p - (l-\alpha)/(n+1)}},$$

then for the total derivatives of the function $\Phi(t, x, v(t, x))$ the estimates

$$\left\{ \iiint_{Q_\tau} \left| \frac{\partial^\alpha \Phi(t, x, v(t, x))}{\partial t^{\alpha_0} \dots \partial x_n^{\alpha_n}} \right|^{\frac{1}{1/p - (l-\alpha)/(n+1)}} dt dx \right\}^{1/p - (l-\alpha)/(n+1)} \leq C A_{Q_\tau} \{B_{Q_\tau}^\alpha + 1\},$$

hold, where C is a constant independent of $v(t, x)$.

This lemma is analogous to the theorem of S. L. Sobolev (see ⁽³⁾, p. 230); its proof is based on the application of the inequalities of Hölder and Minkowski.

Theorem 2. If the coefficients and the right-hand side of equation (3), as functions of the variables (t, x, u) , satisfy in Q_T condition $\widetilde{T}$ with $p = 2$, $l = [(n+1)/2] + 3$, and if there exist $K > 0$, $\delta > 0$, $M_1 > 0$, $M_2 > 0$ such that for the coefficients of equation (3) the inequality

$$Ka^{ij}\xi_i\xi_j + a_t^{ij}\xi_i\xi_j - \delta \left[(b^i - a_{x_j}^{ij})\xi_i \right]^2 - M_1(a_u^{ij}\xi_i\xi_j) - \delta M_2^2 \sum_{j=1}^n (a_u^{ij}\xi_i)^2 \geq 0 \quad (6)$$

is fulfilled for $(t, x) \in Q_T$, $|u| \leq M$, then one can specify $t_0 \leq T$ such that in the cylinder Q_{t_0} there exists a solution of problem (3), (2), continuous with derivatives up to order two inclusive; moreover, $u \in W_2^l(Q_{t_0})$ and

$$|u(t, x)| \leq M, \quad |u_t(t, x)| \leq M_1, \quad |u_{x_i}(t, x)| \leq M_2 \quad (7)$$

in Q_{t_0} . The number t_0 depends on M, M_1, M_2 .

The proof of the theorem is carried out by the method of successive approximations. For the coefficients of the equation

$$L_N(u^N) \equiv -u_{tt}^N + a^{ij}(t, x, u^{N-1})u_{x_i x_j}^N + b^i(t, x, u^{N-1})u_{x_i}^N + b^0(t, x, u^{N-1})u_t^N = f(t, x, u^{N-1}) \quad (8)$$

by virtue of condition $\widetilde{T}$, Lemma 1, and embedding theorems we obtain

$$\max \left\{ \left\| \frac{\partial^\alpha a^{ij}(t, x, u^{N-1}(t, x))}{\partial t^{\alpha_0} \dots \partial x_n^{\alpha_n}} \right\|_{L_q(Q_\tau)}, \left\| \frac{\partial b^i(t, x, u^{N-1}(t, x))}{\partial t^{\alpha_0} \dots \partial x_n^{\alpha_n}} \right\|_{L_q(Q_\tau)}, \right. \\ \left. \left\| \frac{\partial^\alpha f(t, x, u^{N-1}(t, x))}{\partial t^{\alpha_0} \dots \partial x_n^{\alpha_n}} \right\|_{L_q(Q_\tau)} \right\} \leq C \{1 + [U^{(N-1)}(\tau)]^{1/2}\},$$

where

$$\frac{1}{q} = \frac{1}{2} - \frac{l - \alpha}{n + 1}, \quad l = 3 + \left[\frac{n + 1}{2} \right], \quad U^{(N-1)}(\tau) = \sum_{\alpha \leq l} \iiint_{Q_\tau} \left(\frac{\partial^\alpha u^{N-1}}{\partial t^{\alpha_0} \dots \partial x_n^{\alpha_n}} \right)^2 dt dx.$$

Applying the theorem on the complete continuity of the embedding operator $W_2^l(Q_\tau)$ into $C_2(Q_\tau)$ ($l = 3 + [(n+1)/2]$) (see ⁽³⁾, p. 93) and inequalities of the form (5) to the solution $u^N(t, x)$ of problem (8), (2), one can prove that there exists a sufficiently small number $t_1(M, M_1, M_2, M_3) > 0$ such that for $\tau \leq t_1$ equation (8) with conditions (2) defines a mapping of the set of functions $u^{N-1} \in W_2^l(Q_\tau)$, satisfying the initial conditions (2), the inequality

properties (7) and such that $U^{(N-1)}(\tau) \leq M_3$, where $M_3 = \text{const} > 0$, into itself. By virtue of the embedding theorems, this set is compact in $C_2^2(Q_\tau)$. Moreover, considering the equation for the difference $v^N = u^N - u^{N-1}$, one can show that

for $\tau \leq t_0(M, M_1, M_2, M_3, \theta)$ the inequality $\|v^N\|_{L_2(Q_\tau)} \leq \theta \|v^{N-1}\|_{L_2(Q_\tau)}$ will hold, where $\theta < 1$, whence it follows that the sequence $\{u^N\}$ converges in the norm $L_2(Q_\tau)$ for $t_0 \leq t_1$. The limiting function will be the desired solution of problem (3), (2) in the cylinder Q_{t_0} .

II. Boundary-value problem. In the cylinder $Q^T = \{[0, T] \times \Omega\}$, for equation (1) we consider the mixed problem with initial conditions (2) and the boundary condition

$$u(t, x)|_S = 0, \quad (9)$$

where $S = \{[0, T] \times \Sigma\}$, and Σ is the boundary of Ω .

Problem (1), (2), (9) was studied in papers ^(1, 6).

We shall say that the function $f(t, x)$ belongs in Q^T to the class of functions W_0 if $f(t, x)$ is the closure in the norm $W_2^l(Q^T)$ of functions $f_\varepsilon(t, x)$, infinitely differentiable in $\overline{Q^T}$, equal to zero in an ε -neighborhood of Ω , where $l = m + 1 + [(n + 1)/2]$.

Theorem 3. *Suppose that in the cylinder Q^T the assumptions of Theorem 1 concerning the coefficients of equation (1) are satisfied for $l \geq m + 1 + [(n + 1)/2]$. In addition, suppose that*

$$a^{ij}(t, x)\xi_i\xi_j \geq \mu\xi_i\xi_i \quad \text{for } (t, x) \in S$$

and all $\xi = (\xi_1, \dots, \xi_n)$, $\mu = \text{const} > 0$; $f(t, x) \in W_0$, and the boundary Σ of the domain Ω belongs to the class A^{l+2} . Then there exists a solution of problem (1), (2), (9), continuous in Q^T with derivatives up to order $m \geq 2$ inclusive, and $u(t, x) \in W_2^l(Q_T)$.

The basic point of the proof, as in Theorem 1, is the derivation of inequalities of the form (5). We note that in proving these inequalities for domains lying inside Q^T the same methods are applied as in Theorem 1, while for estimates in the boundary strip one uses methods analogous to those set out in (7), Ch. III, § 3, and embedding theorems.

We transform the quasilinear equation (3) to the form

$$-u_{tt} + a^{ij}(t, x, u)u_{x_i x_j} + b^i(t, x, u)u_{x_i} + b^0(t, x, u)u_t + c(t, x, u)u = f(t, x). \quad (10)$$

Theorem 4. *Suppose that for the coefficients of equation (10) in Q^T the assumptions of Theorem 2 are satisfied. In addition, suppose that*

$$a^{ij}(t, x, u)\xi_i\xi_j \geq \mu\xi_i\xi_i \quad \text{for } (t, x) \in S, \quad |u| \leq M \text{ and all } \xi = (\xi_1, \dots, \xi_n),$$

where $\mu = \text{const} > 0$; $f(t, x) \in W_0$, and the boundary $\Sigma \in A^{l+2}$. Then one can specify a $t_1(M, M_1, M_2)$ such that in the cylinder Q^{t_1} there exists a classical solution $u(t, x)$ of problem (10), (2), (9); moreover, $u \in W_2^l(Q^{t_1})$, and the inequalities (7) hold for it.

The proof is carried out by the method of successive approximations, as in Theorem 2, using the a priori inequalities proved in Theorem 3 for the linear equation and embedding theorems.

The author expresses gratitude to O. A. Oleinik for her attention to this work.

Moscow State University
named after M. V. Lomonosov

Received
27 IV 1966

CITED LITERATURE

1. O. A. Oleinik, DAN, **169**, No. 3 (1966).
2. O. A. Oleinik, Mat. sborn., **69**, No. 1, 111 (1966).
3. S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, L., 1950.
4. I. S. Berezin, Mat. sborn., **24**, No. 2, 301 (1949).
5. M. H. Protter, Canad. J. Math., **6**, No. 4, 542 (1954).
6. M. L. Krasnov, Mat. sborn., **49**, No. 1, 29 (1959).
7. O. A. Ladyzhenskaya, *The Mixed Problem for a Hyperbolic Equation*, Moscow, 1953.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.