

SYMMETRIZABLE $\ast$ SPACES AND ε -MAPS

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Abstract

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MATHEMATICS

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SYMMETRIZABLE* SPACES AND ε -MAPS

(Presented by Academician P. S. Aleksandrov on 23 V 1966)

Let X be a symmetrizable space with symmetric d and $\varepsilon > 0$.

Definition 1. A map f of the space X onto a topological space Y will be called a **weak ε -map** if for every $y \in Y$ there is a point $x \in X$ such that

$$f^{-1}y \subseteq O_\varepsilon x = E\{z : z \in X, d(x, z) < \varepsilon\}.$$

Definition 2. A map f of the space X onto Y is called an **ε -map** if for every $y \in Y$ $\text{diam}(f^{-1}y) \leq \varepsilon$.

Theorem. If X is a Hausdorff symmetrizable space and for every n the map $f_n : X \rightarrow Y_n$ is a continuous weak $1/n$ -map, then the natural map $f : X \rightarrow \prod_{n=1}^\infty Y_n$ is an embedding; if for every n f_n is a continuous closed $1/n$ -map, then f is a homeomorphism.

Proof. It is necessary to show that f is one-to-one. Suppose that $x \in X$, $y \in X$ and $fx = fy$, i.e. $f_n x = f_n y$ for every n . Since f_n is a weak $1/n$ -map, for each n there exists a point x_n such that $d(x_n, x) = 1/n$, $d(x_n, y) = 1/n$, i.e. $x = \lim x_n$, $y = \lim_{n \rightarrow \infty} x_n$, which is possible only when $x = y$, since, by assumption, X is Hausdorff. Thus, f is an embedding.

Suppose now that f_n is a closed $1/n$ -map for each n . Let F be a closed subset of X . Put $\Phi = fX \cap (\prod_{n=1}^\infty f_n F)$ and show that $fF = \Phi$. It is easy to see that $fF \subseteq \Phi$, and therefore let $y \in \Phi$. Consequently, $y = fx = \{f_n x\}$ for some $x \in X$, for which $f_n^{-1}(f_n x) \cap F \neq \emptyset$. Let $z_n \in f_n^{-1}(f_n x) \cap F$. Since f_n is a $1/n$ -map, $d(x, z_n) \leq 1/n$. Consequently, $x \in F$ ($x = \lim_{n \rightarrow \infty} z_n$, $z_n \in F$). Thus, fF is closed in fX , since Φ is closed in fX , i.e. f is a homeomorphism of X onto fX .

In what follows, let P denote such a property of topological spaces that: a) if X is a space with property P , then every subspace X_0 is also a space with property P , and b) if for every n X_n is a space with property P , then $\prod_{n=1}^\infty X_n$ is also a space with property P .

Corollary 1. *If X is a Hausdorff symmetrizable space, then X embeds in a space with property P if and only if for every $\varepsilon > 0$ it admits a weak ε -map onto a space with property P .*

* We consider spaces symmetrizable in the sense of A. V. Arhangel'skii (see (2)): a space X is called symmetrizable with symmetric d if: 1) $d(x, y) \geq 0$, $d(x, y) = d(y, x)$ for any $x \in X$, $y \in X$; 2) $d(x, y) = 0$ if and only if $x = y$; 3) if U is an open subset of X and $x \in U$, then there exists $\varepsilon > 0$ such that $O_\varepsilon x \subseteq U$; 4) if $A \subset X$, A is not closed and is nonempty, then there exists $x \in X \setminus A$ such that $d(x, A) = 0$.

In particular, X is compactified onto a metrizable space if and only if, for every $\varepsilon > 0$, it admits a weak ε -mapping onto a metrizable space. Moreover: X has property P if and only if, for every $\varepsilon > 0$, it admits a closed ε -mapping onto a space with property P .

In particular, X is metrizable if and only if, for every $\varepsilon > 0$, it admits a closed ε -mapping onto a metrizable space.

Corollary 2. (J. Ceder). *A symmetrizable paracompact space with the first axiom of countability (a strongly symmetrizable paracompact space) admits a compactification onto a metrizable space.*

The last assertion can be strengthened as follows: if X is a symmetrizable paracompact space with the first axiom of countability and, for each $n = 1, 2, \dots$, ω_n denotes an open cover of the space X , then there exists a metrizable space R and a compactification $f: X$ onto R such that, for each n , f is an ω_n -mapping.

Proof. Put $\eta_n = \{\langle O_{1/n}x \rangle, x \in X\}$. Next we construct a sequence of open covers $\gamma_1, \gamma_2, \dots$ such that:

- 1) γ_1 is star-refined into $\omega_1 \wedge \eta_1$;
- 2) γ_{k+1} is star-refined into $\gamma_k \wedge \omega_{k+1} \wedge \eta_{k+1}$.

On the set X we introduce a symmetric function $D(x, y)$ as follows:

1°. If for two given points x and y there exists no element of the cover γ_1 containing both these points, set $D(x, y) = D(y, x) = 1$.

2°. Suppose case 1° does not occur; then there exists $k \geq 1$ such that both points x and y are contained in some element of the cover γ_k . We show that if $x \neq y$, then among these k there is a largest $k = k(x, y) = k(y, x)$. Indeed, otherwise for every k there would exist points x_k such that $x \in O_{1/k}x_k$, $y \in O_{1/k}x_k$, which is impossible, since X is a Hausdorff space. Put then $D(x, y) = 1/2k(x, y)$ and $D(z, z) = 0$ for every $z \in X$. The symmetry of the function D is obvious; it is also clear that $D(x, y) = 0$ if and only if $x = y$. Consequently, we may consider the space $X; D$, whose topology is defined as follows: a subset $A \subseteq X$ is called closed if and only if from $x \in X$ and $x \notin A$ it follows that $D(x, A) > 0$.

The following property of the function $D(x, y)$ is easily proved: if $D(x, y) \leq$

$1/2^{k+1}$, $D(y, z) \leq 1/2^{k+1}$, then $D(x, z) \leq 1/2^k$. Hence it follows that the function D satisfies the following condition.

Limit axiom. If $\lim_{n \rightarrow \infty} D(x, x_n) = 0$ and $\lim_{n \rightarrow \infty} D(x_n, y_n) = 0$, then $\lim_{n \rightarrow \infty} D(x, y_n) = 0$.

Applying a theorem of A. V. Arhangel'skii⁽³⁾, we obtain that the space $(X; D)$ is strongly symmetrizable with the symmetric D and is metrizable. The identity mapping f of the space X onto the space $(X; D)$ is a compactification, since the relation

$$E\{z : z \in X, D(x, z) < 1/2^k\} = \text{St}_{\gamma_{k+1}} x \quad (1)$$

holds for every point x and for every k , by construction.

It remains to show that the mapping f is an ω_n -mapping for each n . But this follows immediately from the fact that relation (1) holds and that the cover γ_{k+1} is star-refined into ω_{k+1} .

Thus Corollary 2 is proved. If it is applied in the special case when X is bi-compact, we obtain the theorem of B. Nemitskii⁽⁵⁾: *a symmetrizable bicomact space with the first axiom of countability is metrizable*. This is so because a compactification of a bicomact space is a homeomorphism. Similarly, if we take into account a theorem of A. V. Arhangel'skii⁽¹⁾, asserting that a feathered paracompact space which is compactified onto a metrizable space—

...property, is metrizable, we obtain A. V. Arhangel'skii's theorem⁽²⁾: *a symmetrizable feathered paracompact space with the first axiom of countability is metrizable*.

In connection with the preceding, the following proposition is of interest.

Proposition. *If X is a symmetrizable space whose symmetrix d is such that for every point $x \in X$ and for every $\varepsilon > 0$ there is a $\delta = \delta(x, \varepsilon) > 0$ such that from $d(x, y) < \delta$ it follows that $d(x, y) < d(y, X \setminus O_\varepsilon x)$, then X is paracompact with the first axiom of countability.*

Proof. We first show that the symmetrix d agrees with the topology of the space X in the strong sense. For this we prove the following lemma:

Lemma. *If a space Z is symmetrizable with symmetrix ρ , then Z is symmetrizable in the strong sense with symmetrix ρ if and only if ρ satisfies the following condition*

$$(\alpha) \quad \text{If } \lim_{n \rightarrow \infty} \rho(x, x_n) = 0 \text{ and for every } n \quad \lim_{k \rightarrow \infty} \rho(x_n, x_{n,k}) = 0,$$

then there exist sequences of natural numbers $\{m_n\}_{n=1}^\infty$, $\{k_n\}_{n=1}^\infty$

$$\text{such that } \lim_{n \rightarrow \infty} \rho(x, x_{m_n, k_n}) = 0.$$

Indeed, suppose that ρ satisfies condition (α) and let $A \subset Z$. Put $[A]^1 = E\{x : \rho(x, A) = 0\}$. Since ρ satisfies condition (α) , we have $[[A]^1]^1 = E\{x : \rho(x, [A]^1) = 0\} = [A]^1$, and hence $[A]^1$ is closed, i.e. ρ agrees with the topology of Z in the strong sense ($x \in [A]$ if and only if $\rho(x, A) = 0$).

If Z is symmetrizable in the strong sense with symmetrix ρ and $\lim_{n \rightarrow \infty} \rho(x, x_n) = 0$, $\lim_{k \rightarrow \infty} \rho(x_m, x_{m,k}) = 0$, then there exist m_n and k_n such that, for $m \geq m_n$ and $k > k_n$, $x_{m,k} \in (O_{1/n}x)$. Hence $\lim_{n \rightarrow \infty} \rho(x, x_{m_n, k_n}) = 0$, i.e. ρ satisfies condition (α) .

It is easy to verify that the symmetrix d under the assumptions of the proposition satisfies condition (α) .

We now show that the space X is collectively normal. From this, by the Mack-Olin-Bing theorem (see ⁽⁵⁾), our assertion will follow. Let $\{F_\alpha\}$ be a discrete family of closed subsets of the space X .

Put

$$U_\alpha = E\{x : d(x, F_\alpha) < d(x, \bigcup_{\beta \neq \alpha} F_\beta)\}$$

and show that

$$V_\alpha = \langle U_\alpha \rangle \supset F_\alpha.$$

This assertion will prove the claim, since the V_α are pairwise disjoint (because the U_α , obviously, are pairwise disjoint). Let $x \in F_\alpha$, and choose $\varepsilon(x) > 0$ so that

$$O_{\varepsilon(x)}x \cap \left(\bigcup_{\beta \neq \alpha} F_\beta \right) = \Lambda.$$

We have

$$V_\alpha \supset \bigcup_{x \in F_\alpha} \langle O_{\delta(x, \varepsilon(x))}x \rangle \supset F_\alpha,$$

where $\delta(x, \varepsilon(x))$ is chosen according to the condition of the proposition, as was required to prove.

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REFERENCES

- ¹ A. V. Arhangel'skii, *Matem. sborn.*, **67** (109), No. 1 (1965).
- ² A. V. Arhangel'skii, *DAN*, **164**, No. 2, 247 (1965).
- ³ A. V. Arhangel'skii, *UMN*, **4** (130) (1966).
- ⁴ V. Ponomarev, *DAN*, **141**, No. 3 (1961).
- ⁵ F. Michael, *Proc. Am. Math. Soc.*, **9**, No. 5 (1958).
- ⁶ V. Niemytzki, *Math. Ann.*, **104**, H. 5, 666 (1931).
- ⁷ P. S. Aleksandrov, V. V. Nemytskii, *Matem. sborn.*, **3**, No. 3, 663 (1938).

* Therefore, if d is also a Cauchy symmetrix (see ⁽⁷⁾), i.e. if every convergent sequence is fundamental, then X will be metrizable.

Note: Figure translations are in progress. See original paper for figures.

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