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1967

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Abstract

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UDC 518.432

MATHEMATICS

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ON THE ADMISSIBILITY OF PROCEDURES OF THE MONTE CARLO METHOD

(Presented by Academician Yu. V. Linnik on March 18, 1966)

Let $\mathcal{D}$ be a domain of unit volume in k -dimensional Euclidean space; $R = \mathcal{D} \times \mathcal{D} \times \dots \times \mathcal{D}$ —the Cartesian product of $\mathcal{D}$ with itself $n+1$ times; $\Xi = (x_0, \dots, x_n)$, $x_i \in \mathcal{D}$, $i = 0, \dots, n$, and

$$K[f] = \sum_{i=0}^n A_i f(x_i),$$

where f is a function from a given class F of functions defined and integrable in $\mathcal{D}$, and $A_i = A_i(\Xi)$ are determined by specifying a set Q of cubature sums $K[f]$.

Suppose that Ξ is chosen in R at random in accordance with a distribution function $h(\Xi)$, which, for any f from F , with the given $A_i(\Xi)$, satisfies the conditions:

A. The mean value $E_{hK}[f]$ of the random variable $K[f]$ satisfies the equality

$$E_{hK}[f] = \int_{\mathcal{D}} f(x) dx.$$

B. The variance $D_{hK}[f]$ of the random variable $K[f]$ is finite.

If H is the given class of functions h satisfying conditions A and B, then $p = (K[f], h)$ will be called a **procedure of the Monte Carlo method** (M.C.M.) for F in the class of procedures $P = (Q, H)$. If $p' = (K'[f], h')$ ($p' \in P$) is such that $D_{h'}K'[f] \leq D_{hK}[f]$ for any f from F , and in F there is an $\tilde{f}$ for which $D_{h'}K'[\tilde{f}] < D_{hK}[\tilde{f}]$, then we shall say that p' **dominates** p in P for F . We shall call p **inadmissible** in P for F if in P there exists a p' dominating p for F , and **admissible** in the opposite case.

Clearly, the investigation of the admissibility of M.C.M. procedures and the construction of admissible M.C.M. procedures for various P and F is of considerable interest.

Let us consider a certain concrete class of M.C.M. procedures for $f \in L_2$ -functions integrable with square in $\mathcal{D}$. Define $K[f]$ by the equality $K[f] = \Delta(f; \Xi) / \Delta(1; \Xi)$, where $\Delta(f; \Xi) = \det \|f(x_i), \varphi_1(x_i), \dots, \varphi_n(x_i)\|_0^n$, and $\varphi_0(x) \equiv 1, \varphi_1(x), \dots, \varphi_n(x)$ is a finite sequence of orthonormal and almost everywhere bounded functions in $\mathcal{D}$, and we shall assume that Q consists of the single element $K[f]$. Define H as the set of distribution functions $h(\Xi)$ for which conditions A and B are fulfilled and there exists a probability density function, symmetric with respect to the variables $x_0, x_1, \dots, x_n$. We shall call the sequence $\{\varphi_i(x)\}_0^n$ **regular** in $\mathcal{D}$ if $\varphi_j(x)$ are linearly independent on any measurable set d such that $d \subset \mathcal{D}$ and $\text{mes } d \neq 0$, and **irregular** otherwise. In ⁽¹⁾ it was shown that the function $W(\Xi)$, defined by the equality

$$dW(\Xi) = \frac{1}{(n+1)!} \Delta^2(1; \Xi) d\Xi,$$

belongs to H . Below the admissibility of $(K[f], W)$ will be investigated.

Theorem 1. *If $\{\varphi_j(x)\}_0^n$ is regular in $\mathcal{D}$, then the M.C.M. procedure $(K[f], W)$ is inadmissible in P for L_2 .*

Indeed, set

$$d\tilde{h}(\Xi) = \left(1 + \beta \sum_{p < q} \psi(x_p) \psi(x_q) \right) dW(\Xi),$$

where $p, q = 0, 1, \dots, n$; $\psi(x)$ is an almost everywhere bounded function in $\mathfrak{D}$, which is orthogonal to all products $\varphi_r(x) \varphi_s(x)$ for $r, s = 0, 1, \dots, n$, and $\beta > 0$ is a constant satisfying the condition $1 + \beta \sum_{p < q} \psi(x_p) \psi(x_q) \geq 0$ for almost all Ξ in R . Direct calculations show that condition A is satisfied for $\tilde{h}$ and

$$D_{\tilde{h}} K[f] = D_{WK}[f] - \beta \sum_{j=1}^n \left(\int_{\mathfrak{D}} f(x) \psi(x) \varphi_j(x) dx \right)^2,$$

which proves the theorem.

For the special case of a nonregular sequence $\{\varphi_j(x)\}_0^n$ (a sequence of Haar functions), $(K[f], W)$ turns out to be admissible in P for L_2^2 . We define the functions $\varphi_j(x)$ in $\mathfrak{D}$ as follows. Partition $\mathfrak{D}$ into 2^m subdomains $d(m, i_m)$, $i_m = 1, 2, \dots, 2^m$, of equal measure in such a way that, as $m \rightarrow \infty$, the diameter of $d(m, i_m)$ tends to zero. Define the domains $d(s, i_s)$ ($0 \leq s < m$, $i_s = 1, \dots, 2^s$) by the equality $d(s, i_s) = d(s+1, 2i_s - 1) \cup d(s+1, 2i_s)$, and put

$$\chi_{i_s}^s(x) = \begin{cases} 2^{s/2}, & x \in d(s+1, 2i_s-1), \\ -2^{s/2}, & x \in d(s+1, 2i_s), \\ 0, & x \notin d(s, i_s), \end{cases} \quad j = 2^s + i_s - 1,$$

$$\varphi_0(x) \equiv 1, \quad \varphi_j(x) = \chi_{i_s}^s(x) \quad (j = 1, \dots, 2^m - 1). \quad (1)$$

Fix some m_0 , and let $n = 2^{m_0} - 1$.

Theorem 2. For $\{\varphi_j(x)\}_0^n$, defined by equalities (1), the p.m.M.-C. $(K[f], W)$ is admissible in P for $L_{\mathfrak{D}}^2$.

The main stages of the proof are as follows. In our case

$$K[f] = \frac{1}{n+1} \sum_{i=0}^n f(x_i)$$

and the determinant $\Delta(1; \Xi)$ is equal to zero if x_i and x_j for $i \neq j$ belong to one and the same $d(m_0, i_{m_0})$, and is constant if all x_i belong to distinct $d(m_0, i_{m_0})$. For $n = 1$, any $h \in H$ can be defined by the equality

$$dh(\Xi) = \sum_{i,j=0}^{\infty} a_{i,j} \varphi_i(x_0) \varphi_j(x_1) d\Xi.$$

Denote

$$\alpha_j = \int_{\mathfrak{D}} f(x) \varphi_j(x) dx$$

and express $D_{hK}[f]$ through $a_{i,j}$ and α_j . Taking into account that $K[f]$ is defined only for $\Delta(1; \Xi) \neq 0$ and condition A, we obtain

$$D_{hK}[f] = D_{WK}[f] + \sum_{r,s=0}^{\infty} {}'b_{r,s} \alpha_r \alpha_s,$$

where the prime on the summation sign means that the terms corresponding to $r = s$ are absent from the sum, while $b_{r,s}$ are expressed linearly in terms of $a_{i,j}$. Obviously, in $L_{\mathfrak{D}}^2$ there will be a function $\tilde{f}$ for which $D_{hK}[\tilde{f}] > D_{WK}[\tilde{f}]$, which proves the theorem for $n = 1$. The general case reduces to the case $n = 1$, if $\tilde{f}$ is sought among functions different from zero only in one of the domains $d(m_0 - 1, i_{m_0-1})$. As was noted in (2), the indicated procedure dominates the "ordinary" p.m.M.-C.

$$\left(\frac{1}{n+1} \sum_{i=0}^n f(x_i), l \right), \quad \text{where } dl = d\Xi.$$

The author expresses gratitude to A. M. Kagan for valuable comments during discussion of the work.

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Received

4 III 1966

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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