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1967

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Abstract

Full Text

Doklady of the Academy of Sciences of the USSR
1967. Volume 173, No. 5

UDC 513.88:517.948.32

MATHEMATICS

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ON CYCLIC SUBSPACES OF DISSIPATIVE OPERATORS

(Presented by Academician I. N. Vekua, 10 VI 1966)

1. Let $\mathfrak{H}$ be a separable Hilbert space and let A be a bounded linear operator acting in $\mathfrak{H}$. If the operator A is unicellular ⁽¹⁻⁵⁾, then there exists for it a vector $g \in \mathfrak{H}$ such that the linear span of the sequence $g, Ag, A^2g, \dots$ is dense in $\mathfrak{H}$. In the present article, for a certain class of operators, the converse assertion is established.
2. We shall agree to assign an operator A to the class Ω_0 when the following conditions are fulfilled: 1) the entire spectrum of the operator A is concentrated at zero; 2) the operator A is dissipative, i.e. its imaginary component

$$A_I = \frac{1}{2i}(A - A^*)$$

is nonnegative;

$$3) \quad \sigma(A) = \overline{\lim_{\lambda \rightarrow 0} (|\lambda| |\ln |\lambda|| \| (A - \lambda E)^{-1} \|)} < \infty. \quad (1)$$

The class Ω_0 contains, in particular, all Volterra dissipative operators whose imaginary component is nuclear ^(3,6).

If the subspace $\mathfrak{H}_0$ is invariant with respect to an operator $A \in \Omega_0$ and $A_0 = A|_{\mathfrak{H}_0}$ is the restriction of the operator A to the subspace $\mathfrak{H}_0$, then A_0 also belongs to Ω_0 .

3. Let $\mathfrak{G}$ be a separable Hilbert space with scalar product $(f, g)_{\mathfrak{G}}$ ($f, g \in \mathfrak{G}$), and let B be a simple Volterra dissipative operator with one-dimensional imaginary component, defined in $\mathfrak{G}$. Consider the Hilbert space $\tilde{\mathfrak{G}}$, consisting of sequences

$$\tilde{f} = \|f_1, f_2, \dots\| \quad \left(f_k \in \mathfrak{G}, \sum_{k=1}^{\infty} (f_k, f_k)_{\mathfrak{G}} < \infty \right),$$

in which the scalar product is defined by the equality

$$(\tilde{f}, \tilde{g})_{\mathfrak{G}} = \sum_{k=1}^{\infty} (f_k, g_k)_{\mathfrak{G}},$$

and define in $\mathfrak{G}$ the operator $\tilde{B} = \tilde{B}_l$ by the formula

$$\tilde{B}\tilde{f} = \|Bf_1, Bf_2, \dots\| \quad (l = \text{sp } B_l). \quad (2)$$

Obviously, $B \in \Omega_0$ and $\sigma(\tilde{B}) = 2l$.

The operator $\tilde{B}$ has the following universal property*: every simple operator A of the class Ω_0 , for which $\sigma(A) \leq 2l$, is unitarily equivalent to the restriction of the operator $\tilde{B}_l$ to one of its invariant subspaces.

4. Denote by $\mathfrak{G}_{\xi}$, where $\xi = (\xi_1, \xi_2, \dots)$ is an arbitrary orthonormal basis of the Hilbert space $l^{(2)}$, the subspace in $\mathfrak{G}$ consisting of all vectors

* This assertion was originally established by M. S. Brodskii and the author ⁽⁷⁾ (see also ⁽⁸⁾) for the case when the imaginary component of A is nuclear, and then generalized by L. E. Isaev.

of the form $\|\xi_1 f, \xi_2 f, \dots\|$ ($f \in \mathfrak{G}$). The orthoprojector $\tilde{P}_{\xi}$ onto the subspace $\mathfrak{G}_{\xi}$ is defined by the formula

$$\tilde{P}_{\xi}\tilde{f} = \|\xi_1 f, \xi_2 f, \dots\| \quad \left(\tilde{f} = \|f_1, f_2, \dots\|, \quad f = \sum_{k=1}^{\infty} \bar{\xi}_k f_k \right). \quad (3)$$

The orthogonal complement $\mathfrak{G}'_{\xi} = \mathfrak{G} \ominus \mathfrak{G}_{\xi}$, as is easy to see, is the set of vectors of the form

$$\|\varphi_1, \varphi_2, \dots\| \quad \left(\varphi_k \in \mathfrak{G}, \quad \sum_{k=1}^{\infty} \bar{\xi}_k \varphi_k = 0 \right).$$

The following propositions are easily verified.

- I. For any ξ , the subspaces $\mathfrak{G}_{\xi}$ and $\mathfrak{G}'_{\xi}$ are invariant with respect to the operator $\tilde{B}$.
- II. The operator $\tilde{B}_{\xi} = \tilde{B}|_{\mathfrak{G}_{\xi}}$ is simple, Volterra, dissipative; its imaginary component is one-dimensional and its trace is equal to l .
- III. If $\xi^{(1)}, \xi^{(2)}, \dots$ is a complete orthonormal sequence in $l^{(2)}$, then the decomposition

$$\widetilde{\mathfrak{G}} = \widetilde{\mathfrak{G}}_{\xi^{(1)}} \oplus \widetilde{\mathfrak{G}}_{\xi^{(2)}} \oplus \dots \quad (4)$$

is valid (see (7)).

Theorem 1. Let $\widetilde{\mathfrak{G}}_0$ be a subspace invariant with respect to the operator $\widetilde{B}$, and let $\widetilde{B}_0 = \widetilde{B}|_{\widetilde{\mathfrak{G}}_0}$. If the operator $\widetilde{B}_0$ is nonunivalent, then the subspace $\widetilde{\mathfrak{G}}$ has a nonzero intersection with each of the subspaces $\widetilde{\mathfrak{G}}'_\xi$.

5. Consider the Hilbert space $\vec{L} = L^{(2)}(0, l)$ of vector-functions

$$\mathbf{f} = \|f_1(x), f_2(x), \dots\| \quad (f_k(x) \in L^{(2)}(0, l))$$

with scalar product

$$(\mathbf{f}, \mathbf{g}) = \sum_{k=1}^{\infty} \int_0^l f_k(x) \overline{g_k(x)} dx \quad (5)$$

and the integration operator acting in it

$$I\mathbf{f} = \|If_1(x), If_2(x), \dots\| \quad \left(If(x) = 2i \int_0^x f(t) dt \right).$$

According to a theorem of M. S. Livshits ⁽⁹⁾, the operators $\widetilde{B}_\xi^{(k)} = \widetilde{B}|_{\widetilde{\mathfrak{G}}_{\xi^{(k)}}}$ ($k = 1, 2, \dots$) and I are unitarily equivalent. Therefore there exists an isometric mapping U_k of the subspace $\widetilde{\mathfrak{G}}_{\xi^{(k)}}$ onto the subspace $\vec{L}_k = \{\|0, \dots, f_k(x), 0, \dots\|\}$ such that

$$\widetilde{B}_{\xi^{(k)}} = U_k^* I_k U_k \quad (I_k = I|_{\vec{L}_k}, \quad k = 1, 2, \dots),$$

$$\widetilde{B} = U^* I U \quad \left(U = \sum_{k=1}^{\infty} U_k \widetilde{P}_{\xi^{(k)}} \right).$$

6. Let $\mathfrak{H}_0$ be an invariant subspace of the operator A . We shall agree to call the subspace $\mathfrak{H}_0$ **cyclic** with respect to the operator A if there exists a vector $g \in \mathfrak{H}_0$ such that the closure of the linear span of the sequence $A^n g$ ($n = 0, 1, \dots$) coincides with $\mathfrak{H}_0$. In this case we shall call the vector g a **generating vector** of the operator A in the subspace $\mathfrak{H}_0$.

In the paper ⁽²⁾ it was proved that the function $g(x)$ is generating for the operator I in the space $L^{(2)}(0, l)$ if and only if the measure of the set of all $x \in [0, \delta]$ for which $g(x) \neq 0$ is positive for every $\delta > 0$.

Lemma. Let $\mathfrak{H}_1, \mathfrak{H}_2, \dots$ be a sequence of invariant subspaces of a unicellular operator A , and let g_k be a generating vector of the operator A in the subspace $\mathfrak{H}_k$ ($k = 1, 2, \dots$), with

$$\sum_{k=1}^{\infty} (g_k, g_k) < \infty.$$

Then there exists a sequence of numbers $\xi_1, \xi_2, \dots$

$$\left(\sum_{k=1}^{\infty} |\xi_k|^2 = 1 \right),$$

such that the vector

$$g = \sum_{k=1}^{\infty} \xi_k g_k$$

will be generating for the operator A in the subspace

$$\mathfrak{H} = \bigcup_{k=1}^{\infty} \mathfrak{H}_k.$$

Theorem 2. Let A be a simple operator of class Ω_0 , and let $\mathfrak{H}_0$ be its invariant subspace. In order that the subspace $\mathfrak{H}_0$ be cyclic with respect to the operator A , it is necessary and sufficient that the restriction $A_0 = A|_{\mathfrak{H}_0}$ be unicellular.

Proof. Sufficiency is obvious. Let us proceed to establish necessity.

Put $l = \frac{1}{2}\sigma(A_0)$. Using the proposition given in Sec. 3, we find an invariant subspace $\widetilde{\mathfrak{G}}_0$ with respect to the operator $\widetilde{B} = \widetilde{B}_l$ such that the operators A_0 and $\widetilde{B}_0 = \widetilde{B}|_{\widetilde{\mathfrak{G}}_0}$ are unitarily equivalent. By the necessity condition, some vector $\tilde{g} = \|g_1, g_2, \dots\|$ is generating for the operator $\widetilde{B}$ in the subspace $\widetilde{\mathfrak{G}}_0$. Each of the subspaces $\mathfrak{G}_k$, spanned respectively by the vectors $g_k, Bg_k, Bg_k^2, \dots$ ($k = 1, 2, \dots$), is invariant with respect to the operator B . Since the operator B is unicellular ⁽²⁾, applying the lemma we choose a sequence of numbers

$$\xi_1, \xi_2, \dots \left(\sum_{k=1}^{\infty} |\xi_k|^2 = 1 \right)$$

so that the vector

$$g = \bar{\xi}_1 g_1 + \bar{\xi}_2 g_2 + \dots$$

is generating for the operator B in the subspace

$$\mathfrak{G} = \bigcup_{k=1}^{\infty} \mathfrak{G}_k.$$

We shall show that $\mathfrak{G}_0 = \mathfrak{G}$. Indeed, assuming the contrary, we obtain that $\sigma(\tilde{B}_0) < 2l$, and this contradicts the definition of l , in view of the unitary equivalence of the operators $\tilde{B}_0$ and A_0 .

Consider the decomposition (4) of the space $\mathfrak{G}$, where $\xi^{(1)} = (\xi_1, \xi_2, \dots)$, and $\xi^{(2)}, \xi^{(3)}, \dots$ are arbitrary unit vectors supplementing $\xi^{(1)}$ to an orthonormal basis in $l^{(2)}$, and let U be an isometric mapping of the space $\mathfrak{G}$ onto the space $\tilde{L}$, defined in Sec. 5. The vector $U\tilde{g} = \dot{g} = \|g_1(x), g_2(x), \dots\|$ will evidently be generating for the operator I in the subspace $\tilde{L}_0 = U\mathfrak{G}_0$. Since, according to formulas (3) and (6),

$$\tilde{g}^{(1)} = \tilde{P}_{\xi^{(1)}} \tilde{g} = \|\xi_1 g, \xi_2 g, \dots\|$$

and the vector g is generating for the operator B in $\mathfrak{G}$, it follows that $\tilde{g}^{(1)}$ is a generating vector of the operator $\tilde{B}$ in $\mathfrak{G}_{\xi^{(1)}}$. Therefore the function $g_1(x)$ will be generating for the operator I in the space $L^{(2)}(0, l)$.

Denote by $\mathfrak{M}$ the linear span of the sequence

$$I^n g = \|I^n g_1(x), I^n g_2(x), \dots\| \quad (n = 0, 1, \dots).$$

Since

$$I^n g(x) = \frac{(2i)^n}{(n-1)!} \int_0^x (x-t)^{n-1} g(t) dt = \frac{(2i)^n}{(n-1)!} \int_0^x t^{n-1} g(x-t) dt \quad (n = 1, 2, \dots),$$

then $\mathfrak{M}$ is the linear span of the vectors $\mathbf{g}$ and $\|p * g_1, p * g_2, \dots\|$, where $p * g$ denotes the convolution of the functions $p(x)$ and $g(x)$, while $p(x)$ ranges over the set of all polynomials ($0 \leq x \leq l$). We shall show that the intersection of the subspaces $\mathfrak{M} = \tilde{L}_0$ and $\tilde{L}'_1 = \tilde{L} \ominus \tilde{L}_1 = \{\|0, f_2(x), f_3(x), \dots\|\}$ consists of the zero vector. To this end suppose that

$$(\lambda_n g_1 + p_n * g_1) \rightarrow 0, \quad (\lambda_n g_k + p_n * g_k) \rightarrow h_k \quad (k = 2, 3, \dots) \quad (7)$$

in the sense of convergence in $L^{(2)}(0, l)$. Using the known properties of convolution ⁽¹⁰⁾ and the equalities (7), we obtain $g_1 * h_k = 0$ ($k = 2, 3, \dots$). Taking into account that the function $g_1(x)$ is generating for the integration operator in

$L^{(2)}(0, l)$, and applying Titchmarsh' s theorem ⁽¹¹⁾, we arrive at the conclusion that $h_k = 0$ ($k = 2, 3, \dots$). Thus, by Theorem 1, the operator I_0 is unicellular and, consequently, the operator A_0 is also unicellular.

Let us note that without the requirement of dissipativity the necessity in Theorem 2 is, generally speaking, false. Indeed, the operator $\mathbf{J} = \|If_1(x), -If_2(x)\|$, acting in the space $\tilde{L} = \{\|f_1(x), f_2(x)\|\} (f_k(x) \in L^{(2)}(0, l))$, is not unicellular. However, if the function $g(x)$ is generating for the operator I in $L^{(2)}(0, l)$, then the vector-function $\mathbf{g} = \|g(x), g(x)\|$ will be generating for the operator $\mathbf{J}$ in $\tilde{L}$.

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Received
8 VI 1966

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Note: Figure translations are in progress. See original paper for figures.

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