

ON A TRANSFORMATION OF OPERATOR SUMS

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Abstract

Full Text

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MATHEMATICS

S. I. OSIPOV

ON A TRANSFORMATION OF OPERATOR SUMS

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Let $D = d/dt$ be the differentiation operator, $\theta_\alpha = t^\alpha D$, α any complex number,* and

$$\theta_{\alpha_k} \theta_{\alpha_{k-1}} \cdots \theta_{\alpha_1} \quad (1)$$

denote the superposition of the operators $\theta_{\alpha_1}, \theta_{\alpha_2}, \dots, \theta_{\alpha_k}$ ($k = 1, 2, \dots$), and let G_n be the set of all functions of t differentiable n times.

The operator sum

$$\sum_{p=1}^n a_{n-p} (\theta_{\alpha_k} \theta_{\alpha_{k-1}} \cdots \theta_{\alpha_1})^p \quad (2)$$

with arbitrary constant coefficients a_{n-p} can be transformed to the form

$$\sum_{\nu=1}^{nk} Q(n, k, \nu, \alpha_i, t) D^\nu, \quad (3)$$

where $Q(n, k, \nu, \alpha_i, t)$ are certain completely determined functions. Let

$$Q(n, k, \nu, \alpha_i, t) = \sum_{p=\mathbb{E}(\frac{\nu+k-1}{k})}^n C(k, \nu, p, \alpha_i) a_{n-p} t^{-\mu\nu}, \quad (4)$$

where

$$\mu = k - \sum_{i=1}^k \alpha_i,$$

$$C(k, \nu, p, \alpha_i) = \frac{1}{(\nu!)^2} \sum_{m=\nu}^{pk} \frac{m!}{(m-\nu)!} S(k, p, m, \alpha_i),$$

$$S(k, p, m, \alpha_i) = \sum_{r=0}^m \frac{(-1)^{r+pk} m!}{r!(m-r)!} N(p, k, \alpha_i, r),$$

$$N(p, k, \alpha_i, r) = \prod_{j=1}^p R(p, k, \alpha_i, r, j),$$

$$R(p, k, \alpha_i, r, j) = [r + (p-j+1)\mu + \alpha_k][r + (p-j+1)\mu + \alpha_k + \alpha_{k-1} - 1] \\ \times \cdots \times [r + (p-j+1)\mu + \alpha_k + \alpha_{k-1} + \cdots + \alpha_1 - (k-1)],$$

$E\left(\frac{\nu+k-1}{k}\right)$ is the integer part of the number $\frac{\nu+k-1}{k}$.

Theorem. On the set G_{nk} , the operators (2) and (3)–(4) are equivalent.

The proof is obtained by generalizing our method ⁽¹⁾.

* The function t^α , as a general power function, is generally multivalued, and it is necessary to fix some single-valued branch of it.

Relying on the theorem, one can solve the following problems.

Problem 1. A linear differential operator is given,

$$q_0(t)D^s + q_1(t)D^{s-1} + \cdots + q_{s-1}(t)D,$$

where $q_0(t), q_1(t), \dots, q_{s-1}(t)$ are linear combinations of generalized power functions. It is required to determine the possibility (or impossibility) of representing it in the form of a polynomial in the operator (1).

This problem is of interest in connection with methods for integrating ordinary linear differential equations supplied by the operational calculus of V. A. Ditkin ⁽²⁾ and its generalizations.

Problem 2. Let σ be any complex number,

$$\delta(n, m) = \begin{cases} 0, & \text{if } n < m, \\ 1, & \text{if } n \geq m, \end{cases}$$

for $p = 1, 2, \dots$ and $\xi = 1, 2, \dots, (p+1)k$ the numbers $A_{\xi, p}$ satisfy the recurrence equation

$$A_{\xi,p+1} = \sum_{\nu=1}^{\min(\xi,pk)} \sum_{r=1}^k \frac{\delta(\nu+r, \xi) r! \Gamma(\nu-p\sigma+1)}{(\nu+r-\xi)! (\xi-\nu)! \Gamma(\xi-p\sigma-r+1)} A_{r,1} A_{\nu,p}$$

under the conditions $A_{r,1} = c_r$, where the c_r are arbitrarily prescribed constants with the sole restriction $c_k \neq 0$, and it is required to find $A_{\xi,p}$ in the form of some analytic expression.

In the solution of this problem an important role is played by $C(k, \nu, p, \alpha_i)$.

The Lah numbers ⁽³⁾ and the classical Stirling numbers of the second kind are special cases of $A_{\xi,p}$.

In conclusion, let us note the formula

$$(xy)^{-\lambda/2} e^{-y} I_{\lambda}(2\sqrt{xy}) = \sum_{k=0}^n \frac{L_k^{\lambda}(x)}{\Gamma(\lambda+k+1)} (-y)^k + \frac{(-y)^{n+1}}{\Gamma(\lambda+n+2)} e^{x-\xi^2} L_{n+1}^{\lambda}(\xi^2), \quad (5)$$

$$x \geq y \geq 0, \quad \lambda \geq -\frac{1}{2}, \quad \xi = \sqrt{x} + \theta\sqrt{y}, \quad |\theta| < 1.$$

It can be obtained by relying on a result of B. M. Levitan ⁽⁴⁾ and on our representation ⁽⁵⁾ of the Laguerre polynomials $L_n^{\lambda}(x)$ by means of the operator (1) for $k=2$, $\alpha_1 = 1 + \lambda$, $\alpha_2 = -\lambda$. Formula (5) is closely related to the well-known formula of N. Ya. Sonin ⁽⁶⁾ and differs by the remainder term.

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Note: Figure translations are in progress. See original paper for figures.

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