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Abstract

Full Text

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MATHEMATICS

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ON SOME MULTIDIMENSIONAL INTEGRAL OPERATORS WITH HOMOGENEOUS KERNELS

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In the present paper we study the properties of linearity and complete continuity of certain classes of integral operators with kernels homogeneous of order $-n$, where n is the dimension of the integral. In the one-dimensional case their linearity in L^p was established by Hardy ⁽³⁾, and in a number of spaces of types M and C —in ⁽⁴⁾; a number of theorems on the solvability of integral equations was also obtained ⁽²⁾. Multidimensional operators with kernels of the form $T(x, y) = |x|^{-\alpha}|x - y|^{\alpha-n}$, $0 < \alpha < n$, were studied in ⁽¹⁾. Such operators arise from differential equations with singular coefficients.

Let x and y be points of the n -dimensional Euclidean space E_n . A measurable function $\theta(x, y)$ is called homogeneous of order $-n$ if, for all $t > 0$,

$$\theta(tx, ty) = t^{-n}\theta(x, y). \quad (1)$$

We pose the problem of investigating the operators

$$\theta\varphi = \int_D \theta(x, y)\varphi(y) dy, \quad x \in D, \quad (2)$$

where D is an arbitrary bounded or unbounded domain in E_n containing the origin.

Adopting the usual notation for the Banach spaces L^p , M , C , we shall also consider the subclass M^0 (M^∞) of functions from M that are continuous at the point $x = 0$ ($x = \infty$). If $\Phi(x)$ belongs to one of the indicated classes, then by L_β^p , M_β , M_β^0 (M_β^∞), and C_β we shall denote the corresponding isometric Banach spaces of functions (for more details see ⁽¹⁾)

$$\varphi(x) = |x|^{-\beta}\Phi(x), \quad \|\varphi\|_\beta = \|\Phi\|. \quad (3)$$

Theorem 1. Let $\theta(x, y)$ be homogeneous of order $-n$, satisfy the summability condition*

$$Q(\beta) = \int_{E_n} |\theta(j, u)| |u|^{-\beta} du < +\infty, \quad j = (1, 0, \dots, 0), \quad (4)$$

and be invariant under arbitrary rotations γ of the space E_n in the sense that

$$\theta[\gamma(x), \gamma(y)] = \theta(x, y). \quad (5)$$

Then the operator $\theta\varphi$ is linear in $M_\beta(D)$, and if $x = 0$ ($x = \infty$) is an interior point of D , then also in $M_\beta^0(D)$ ($M_\beta^\infty(D)$). If neither of the points $x = 0$, $x = \infty$ is a boundary point for D , then $\theta\varphi$ is linear in $C_\beta(D)$. If, finally, in addition to (4), the condition

$$Q_1(\beta) = \int_{E_n} |\theta(v, j)| |u|^{\beta-n} du < +\infty, \quad (6)$$

is satisfied, then the operator is linear in $L_{\beta-n/p}^p(D)$ for every $p \geq 1$.

* Multidimensional singular integrals ⁽⁵⁾ do not satisfy this condition.

Proof.

- 1) **Linearity in M_β .** Introducing the function $\Phi(x)$ in accordance with (3) and putting $\theta\varphi = |x|^{-\beta}\Omega(x)$, we have

$$\Omega(x) = \int_D \left\{ \frac{|x|}{|y|} \right\}^\beta \theta(x, y) \Phi(y) dy. \quad (7)$$

Let $u = \gamma_x^{-1}(y)$ be such a rotation of the coordinate system $oy_1y_2 \dots y_n$ after which the first axis oy_1 lies on the straight line ox , and let $y = \gamma_x(u)$ be the inverse rotation; let $u = \delta_x(v)$ be the similarity transformation $u_i = |x|v_i$, $i = 1, 2, \dots, n$, and $y = l_x(v)$, $l_x = \gamma_x\delta_x$, their composition. Changing variables $y = l_x(v)$ and using (5) and (1), we have

$$\Omega(x) = \int_{D_x} |v|^{-\beta} \theta(j, v) \Phi[l_x(v)] dv, \quad (8)$$

where $D_x = l_x^{-1}(D)$ and $\text{mes } D_x = |x|^{-n} \text{mes } D$. Hence it follows easily that $\|\Omega\|_M \leq Q(\beta) \|\Phi\|_M$, or $\|\theta\|_{M_\beta} \leq Q(\beta)$, and our first assertion is proved.

- 2) **Linearity in M_β^0 .** If $x = 0$ is an interior point of the domain D , then as $x \rightarrow 0$, $D_x \rightarrow E_n$.

Applying to (8) the procedure of subtracting $\Phi(0)$, it is not difficult to show that $\lim_{x \rightarrow 0} \Omega(x) = \Omega(0)$ exists and

$$\Omega(0) = q(\beta)\Phi(0), \quad q(\beta) = \int_{E_n} |v|^{-\beta} \theta(j, v) dv. \quad (9)$$

An analogous property belongs to the point $x = \infty$, if it is interior.

3) **Linearity in C_β .** Let $x \rightarrow x_0$, $x_0 \in \overline{D}$, and $x_0 \neq 0, \infty$. Then

$$\Omega(x) - \Omega(x_0) = \Omega_1(x) + \Omega_2(x),$$

where

$$\Omega_1(x) = \left(\int_{D_x} - \int_{D_{x_0}} \right) |v|^{-\beta} \theta(j, v) \Phi[l_x(v)] dv,$$

$$\Omega_2(x) = \int_{D_{x_0}} |v|^{-\beta} \theta(j, v) \{ \Phi[l_x(v)] - \Phi[l_x(v_0)] \} dv.$$

Since $\lim_{x \rightarrow x_0} \Phi[l_x(v)] = \Phi[l_x(v_0)]$, we have $\lim_{x \rightarrow x_0} \Omega_2(x) = 0$. Passing to $\Omega_1(x)$, we note that

$$\int_{D_x} - \int_{D_{x_0}} = \int_{(D_x - D_{x_0}) + (D_{x_0} - D_x)}.$$

Assuming additionally that the boundary of the domain D has zero measure, one can prove that as $x \rightarrow x_0$, $\text{mes}(D_x - D_{x_0}) \rightarrow 0$ and $\text{mes}(D_{x_0} - D_x) \rightarrow 0$. Therefore $\lim_{x \rightarrow x_0} \Omega_1(x) = 0$, and the continuity of $\Omega(x)$ is proved.

4) **Linearity in $L^p_{\beta-n/p}$.** Putting $\varphi(x) = |x|^{n/p-\beta} \Phi(x)$, $\theta\varphi = |x|^{n/p-\beta} \Omega(x)$, for $\Omega(x)$ we obtain the form (7) with exponent $\beta - n/p$ instead of β . Applying Hölder's inequality, raising to the power p and integrating, we obtain

$$\|\Omega\|_{L^p} \leq Q^{1/p'}(\beta) Q_1^{1/p}(\beta) \|\Phi\|_{L^p}, \quad \text{or} \quad \|\theta\|_{L^p_{\beta-n/p}} \leq Q^{1/p'}(\beta) Q_1^{1/p}(\beta).$$

The theorem is proved.

Remark 1. If $x = 0$ is a boundary point of the domain D , then the operator does not act in $M_\beta^0(D)$ and $C_\beta(D)$. An analogous role of a special point is played by $x = \infty$, if the domain D is unbounded.

A large class of kernels satisfying condition (5) is constituted by

$$\theta(x, y) = h\{|x|, |y|, |x - y|, |x + y|, (x, y/|y|)\},$$

where h denotes an arbitrary function of 5 variables satisfying the summability condition (4). If, moreover, h does not change under interchange of the first two arguments, then (6) can be reduced to (4) by the transformation with inverse radius-vectors and, thus, linearity in $L_{\beta-n/p}^p$ will be ensured by only one summability condition. Kernels of this type include, for example, the previously studied $T(x, y)$ ¹ (except for cases where special manifolds are present) and the kernel $\theta_0(x, y) = \rho^{-n}$, where $\rho^2 = |x|^2 + |y|^2$. Condition (4) for θ_0 will be fulfilled for all β , $0 < \beta < n$.

Relying on this, we prove

Theorem 2. *Let $P(x, y)$ be homogeneous of degree n , positive definite, i.e. $P(x, y) > 0$ for all x, y for which $\rho^2 = |x|^2 + |y|^2 > 0$, and continuous for the same x, y . Then the operator*

$$P\varphi = \int_D \frac{\varphi(y) dy}{P(x, y)}$$

is linear in $M_\beta(D)$ and $L_{\beta-n/p}^p(D)$ for all β , $0 < \beta < n$, and $p \geq 1$.

Theorem 3. *Let the singular point $x = 0$ be interior. If, in addition to the assumptions of Theorem 1, it is assumed that $q(\beta) \neq 0$, where $q(\beta)$ is given by formula (9), then the operator $\theta\varphi$ will not be completely continuous in M_β , M_β^0 , and C_β . It will not be completely continuous in $L_{\beta-n/p}^p$, $p \geq 1$, if it is assumed that $q_1(\beta) \neq 0$, where*

$$q_1(\beta) = \int_{|u| \leq 1} |u|^{-\beta} \theta(j, u) du.$$

The proof of Theorem 3 in M_β , M_β^0 , and C_β is based on property (9) and is analogous to the proofs of the corresponding theorems in¹, while for $L_{\beta-n/p}^p$ it is analogous to⁶. It is possible that the theorem is true without any assumptions of the type $q(\beta) \neq 0$, as was the case in the one-dimensional case⁴ (see also⁷).

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