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Abstract

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MATHEMATICS

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ESTIMATES OF SOLUTIONS OF GENERAL BOUNDARY-VALUE PROBLEMS FOR A HIGHER-ORDER ELLIPTIC EQUATION IN THE SPACES $C_{l+\alpha(r)}$

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In [1] sharp estimates were obtained for the derivatives of solutions of the Dirichlet problem for a second-order elliptic equation, under the condition that the coefficients and the right-hand side of the equation, as well as the second derivatives of the boundary function, had a modulus of continuity not exceeding a certain function $\omega(r)$ satisfying Dini's condition. It was proved that the second derivatives of the solution have a modulus of continuity not exceeding

$$C \int_0^r \frac{\omega(t)}{t} dt.$$

On fundamental solutions see [4, 5].

Theorems 1 and 4 contain sharp interior estimates for solutions of elliptic and parabolic systems.

In Theorem 2, estimates are obtained for solutions of general boundary-value problems in a half-ball, on the flat part of whose boundary general boundary conditions are prescribed. With the aid of Theorems 1 and 2 we shall obtain estimates for solutions of general boundary-value problems for an arbitrary domain with sufficiently smooth boundary. In Hölder norms these estimates were obtained in [2, 3].

Let us introduce the following definitions: we shall say that a function $f(x)$ satisfies a uniform Hölder condition in the domain D with refined exponent $\alpha(r)$ if, for any points $x, x' \in D$,

$$|f(x) - f(x')| \leq C|x - x'|^{\alpha(|x - x'|)};$$

the function $f(x)$ satisfies a local Hölder condition with refined exponent $\alpha(r)$ if it satisfies a uniform Hölder condition with refined exponent $\alpha(r)$ in every

subdomain D' of the domain D that is strictly interior with respect to D , where the function $\alpha(r)$ is defined and continuous for $0 < r < \infty$ and satisfies the following conditions: $\alpha(r) \rightarrow \lambda$, $\lambda \in [0, 1)$, $\alpha'(r)r \ln r \rightarrow 0$ both as $r \rightarrow 0$ and as $r \rightarrow \infty$, and, if $\lambda = 0$, then $\alpha(r) \ln r \rightarrow -\infty$ as $r \rightarrow 0$, and $\alpha(r) + r \ln r \cdot \alpha'(r) > 0$ for $r \in R_0 = (0, r_0) + (1/r_0, \infty)$, where r_0 is some sufficiently small number. (It is assumed that $\alpha'(r)$ exists and is continuous, at least for sufficiently small and sufficiently large r .) From these conditions it follows that the function $r^{\alpha(r)}$ increases monotonically for $r \in R_0$.

The function $r^{\alpha(r)-\lambda}$ is slowly varying, i.e.

$$(kr)^{\alpha(kr)-\lambda} / r^{\alpha(r)-\lambda} \rightarrow 1$$

as $r \rightarrow 0$ and as $r \rightarrow \infty$, uniformly with respect to k , $0 < a \leq k \leq b < \infty$.

We denote by $C^{\alpha(r)}(D)$ the class of functions satisfying a local Hölder condition with refined exponent $\alpha(r)$ in the domain D . By $C^{l+\alpha(r)}(D)$ we denote the class of functions for which $D^l f(x) \in C^{\alpha(r)}(D)$.

By $C_{l+\alpha(r)}(D)$ we denote the space of functions for which the norm is finite:

$$|f|_{l+\alpha(r)}^D = \sum_{k=0}^l \sup_{\substack{x \in D \\ |j|=k}} |D^j f(x)| + \sup_{\substack{x, x' \in D \\ |j|=l}} \frac{|D^j f(x) - D^j f(x')|}{|x - x'|^{\alpha(|x-x'|)}} \equiv \sum_{k=0}^l [f]_k + [f]_{l+\alpha(r)}.$$

By $C_{p+\beta(r), l+\alpha(r)}(D)$ we mean the space of functions for which the norm

$$\begin{aligned} |f|_{p+\beta(r), l+\alpha(r)}^D &= \sum_{k=0}^l \sup_{\substack{x \in D \\ |j|=k}} d_x^{p+k} |D^j f(x)| + \\ &+ \sup_{\substack{x, x' \in D \\ |j|=l}} d_{xx'}^{p+l+\beta(d_{xx'})+\alpha(d_{xx'})} \frac{|D^j f(x) - D^j f(x')|}{|x - x'|^{\alpha(|x-x'|)}} \equiv \sum_{k=0}^l [f]_{p,k} + [f]_{p+\beta(r), l+\alpha(r)}, \end{aligned}$$

is finite, where d_x is the distance from the point x to the boundary of the domain D and $d_{xx'} = \min(d_x, d_{x'})$, $p + l \geq 0$, p is an integer, $\beta(r)$ is some function, $\alpha(r) + \beta(r) \geq 0$.

By $\mathcal{A}_l$ we denote the class of refined exponents $\alpha(r)$ for which $A^l \alpha(r) < \infty$, where

$$A\alpha(r) = \int_0^r r^{\alpha(r)-1} dr, \quad l = 1, 2, \dots$$

Let $\alpha(r) \in \mathcal{A}_1$ and let

$$B\alpha(r) = \frac{1}{\ln r} \ln \frac{A\alpha(r)}{A\alpha(1)}.$$

Theorem 1. Let the functions $u_j(x) \in C_j^t(D)$ satisfy the elliptic system in the sense of Douglis-Nirenberg

$$\sum_{j=1}^N l_{ij}(x, D) u_j(x) = f_i(x), \quad i = 1, 2, \dots, N, \quad (1)$$

$$l_{ij}(x, D) = \sum_{|p| \leq s_i + t_j} a_{ij,p}(x) D^p, \quad s_i, t_i \text{ are integers, } \max s_i = 0.$$

Suppose that

$$|a_{ij,p}|_{-\alpha(r), -s_i + l + \alpha(r)}, \quad |p| = s_i + t_j,$$

$$|a_{ij,p}|_{s_i + t_j - |p| + B\alpha(r) - \alpha(r), -s_i + l + \alpha(r)}, \quad |p| < s_i + t_j,$$

are bounded by the constant K_1 ; $\alpha(r) \in \mathcal{A}_1$, $l \geq 0$ is an integer. Then

$$\begin{aligned} & |u_j|_{t-t_j, t_j+l+B\alpha(r)} \leq \\ & \leq K_2 \left(\sup_i |f_i|_{s_i+t+B\alpha(r)-\alpha(r), -s_i+l+\alpha(r)} + \sum_{t_j>0} |u_j|_{t-t_j, 0} \right), \quad t = \max_i t_i, \quad (2) \end{aligned}$$

where K_2 depends on K_1 , n , the ellipticity constant and on the function $A\alpha(r)$, the domain D , s_i and t_i .

In the proof the following lemma is used.

Lemma 1. Let

$$u(x) = \int_S D_y^{2m} \Gamma(x-y, a(x)) (g(y) - g(x)) h(y) dy,$$

where $S = \{|x_0 - x| \leq d, d \leq 1\}$, $\Gamma(x, a(\xi))$ is the fundamental solution of the elliptic equation with constant coefficients

$$\sum_{|k|=2m} a_k(\xi) D_x^k u(x) = 0.$$

Let $g(x) \in C_{\alpha(r)}(S)$, $h(x) \in C_{\gamma(r)}(S)$, where $\alpha(r) \in \mathcal{A}_1$,

$$\gamma(r) = \frac{1}{\ln r} \ln \left(r^{B\alpha(r)} + r^{\alpha(r)} \ln \frac{1}{r} \right)$$

and let

$$[a]_{\alpha(r)}^S = \sum_{|k|=2m} [a_k]_{\alpha(r)}^S < \infty.$$

Then the estimate holds

$$|u(x_0) - u(x)| \leq C[g]_{\alpha(r)}^S (1 + [a]_{\alpha(r)}^S) |h|_{\gamma(r)}^S |x_0 - x|^{B\alpha(|x_0 - x|)}$$

where the constant C depends on n , $A\alpha(r)$, and the ellipticity constant.

Let now Σ_R be a half-ball in the space E_{n+1} , $\Sigma_R = \{|x| < R, x_{n+1} \geq 0\}$, $R \leq 1$. Consider in Σ_R the problem

$$L(x, D)u(x) = f(x) \quad (3)$$

$$B_j(x', D)u(x)|_{x_{n+1}=0} = \varphi_j(x), \quad x' = (x_1, x_2, \dots, x_n), \quad j = 1, \dots, m,$$

and suppose: 1) the operators L and B_j satisfy conditions i) and ii) of [3], p. 70 (the leading coefficients of L do not depend on x_{n+1} if $\lambda = 0$); 2) the coefficients of the operators L and B_j are bounded by the constant K_3 in the norms $|\cdot|_{l-2m+\alpha(r)}$ and $|\cdot|_{l-m_j+B\alpha(r)}$, $l \geq l_0 = \max(2m, m_j)$, respectively, $\alpha(r) \in A_4$.

Theorem 2. Let the function $u(x) \in C^{l_0+B^2\alpha(r)}(\Sigma_R)$ be a solution of problem (3). Let assumptions 1), 2) be fulfilled. Suppose that the norms

$$|\bar{f}|_{2m+B^2\alpha(r)-\alpha(r), l-2m+\alpha(r)}, \quad |\varphi_j|_{m_j+B^2\alpha(r)-B\alpha(r), l-m_j+B\alpha(r)}$$

and $|u|_0$ are finite, $\alpha(r) \in A_4$. Then the estimate holds

$$|\bar{u}|_{0, l+B^2\alpha(r)} \leq K_4 \left(|\bar{f}|_{2m+B^2\alpha(r)-\alpha(r), l-2m+\alpha(r)} + \sum_{j=1}^m |\varphi_j|_{m_j+B^2\alpha(r)-B\alpha(r), l-m_j+B\alpha(r)} + |u|_0 \right), \quad (4)$$

where K_4 depends on n , on the function $A^4\alpha(r)$, K_3 , and on the constants entering condition 1). (The bar over the norm means that the distance entering its definition is measured only up to the curvilinear boundary Σ_R .)

Estimate (4) is proved from a certain representation for the derivatives $D^l u(x)$ and with the aid of Lemma 2.

Lemma 2. Let

$$u(x) = \int K(x' - y', x_{n+1}) f(y') dy', \quad x_{n+1} > 0,$$

where

$$K(x) = W(x/|x|)|x|^{-n},$$

$W(Q)$ is a continuously differentiable function on the half-sphere $|Q| = 1$, $x_{n+1} \geq 0$, and

$$\int_{|\xi|=1} W(\xi', 0) d\Omega_\xi = 0.$$

Let $f(x') \in C_{\alpha(r)}(E_n)$, $\alpha(r) \in A_1$, and let $f(x') \in L_p$ for some finite $p \geq 1$. Then

$$[u]_{B\alpha(r)} \leq C[f]_{\alpha(r)},$$

where C depends on n , $\max |W|$, $\max |DW|$, and $A\alpha(r)$.

Lemma 3. Let

$$u(x) = \int K(x' - y', x_{n+1}) \int D^{2m} \Gamma(y' - \eta', -\eta_{n+1}) (g(\eta) - g(x'_0, 0)) h(\eta) d\eta dy',$$

$$x_{n+1} > 0,$$

where the kernel $K(x)$ satisfies the conditions of Lemma 2, while $g(x) \in C_{\alpha(r)}(E_{n+1})$, $h(x) \in C_{B^2\alpha(r)}(E_{n+1})$ and is finite, and the function $\Gamma(x)$ is the fundamental solution of an elliptic equation with constant coefficients

$$\sum_{|k|=2m} a_k D^k u(x) = 0, \quad \alpha(r) \in A_4.$$

Then

$$|u(x_0) - u(x)| \leq C[g]_{\alpha(r)} \left(|h|_0 + |x_0 - x|^{B^4\alpha(|x_0-x|)} [h]_{B^2\alpha(r)} \right) |x_0 - x|^{B^2\alpha(|x_0-x|)},$$

where the constant C depends on n , $\max |W|$, $\max |DW|$, and on the function $A^4\alpha(r)$ and the constant of ellipticity.

Let now D be an arbitrary domain, which may be unbounded, in the space E_{n+1} . We shall denote its boundary by ∂D . Consider in the domain D the problem

$$Lu(x) = f(x), \quad B_j u(x)|_\Gamma = \varphi_j(x), \quad j = 1, \dots, m, \quad (5)$$

where Γ is a part of the boundary ∂D , which may coincide with the entire boundary of the domain D ; L and B_j are differential operators of orders $2m$ and m_j . Put $l_0 = \max(2m, m_j)$ and let $l \geq l_0$ be some integer and $\alpha(r) \in A_4$.

Consider a subdomain (possibly unbounded) U of the domain D , such that $\partial U \cap \partial D \subset \Gamma$. We shall assume that $\Gamma \in C^{l+\alpha(r)}$. Suppose that for U and D the conditions of [3], p. 76, are fulfilled, and it is assumed that each component of the mapping T_P (see [3], p. 76) and its inverse has norm $|\cdot|_{l+\alpha(r)}$ bounded by the constant K_5 , independent of P . With respect to the equation and the boundary conditions we shall assume that, under any such mapping T_P , they pass into system (3) in the half-ball $\Sigma_{R(P)}$, satisfying conditions 1), 2). -

assume that the coefficients of the operators L and B_j have, respectively in the domain D , finite K_6 -norms $|\cdot|_{l-2m+\alpha(r)}$ and $|\cdot|_{l-m_j+B\alpha(r)}$, and that the operator L is uniformly elliptic in this domain.

Let $|f|_{l-2m+\alpha(r)}^D$, $|\varphi_j|_{l-m_j+B\alpha(r)}^\Gamma$, and $|u|_0^D$ be finite, $\alpha(r) \in A_4$.

Theorem 3. Let the function $u(x) \in C^{l+B^2\alpha(r)}$ in $D + \Gamma$ be a solution of problem (5). Then

$$u(x) \in C^{l+B^2\alpha(r)} \quad \text{in } \bar{U},$$

$$|u|_{l+B^2\alpha(r)}^U \leq K_7 \left(|f|_{l-2m+\alpha(r)}^D + \sum |\varphi_j|_{l-m_j+B\alpha(r)}^\Gamma + |u|_0^D \right),$$

where the constant K_7 depends on d , K_5 , m , m_j , K_6 , on the function $A^4\alpha(r)$, and on the constants entering condition 1).

If the domain D is bounded, then in Theorem 3 one may take $U = D$ and $\Gamma = \partial D$. In this case the normal solvability of problem (5) follows from Theorem 3.

For $\alpha(r) \in A_1$, interior estimates have been obtained for parabolic systems in the sense of Petrovsky (for second order, see ⁽⁶⁾). For brevity we formulate the result for a single equation. Let D be a bounded domain in $(n+1)$ -dimensional space $(x_1, x_2, \dots, x_n, t)$. Introduce the distance

$$d(P, Q) = \sqrt{|x - \xi|^2 + |t - t'|^{1/m}}. \quad (*)$$

Let

$$|f|_{p+3(r), \alpha(r)} = \sup_{P \in D} d_P^p |f(P)| + \sup_{P, Q \in D} d_{PQ}^{p+\alpha(d_{PQ})+\ell(d_{PQ})} \frac{|f(P) - f(Q)|}{d(P, Q)^{\alpha(d(P, Q))}},$$

where $p \geq 0$ is an integer, d_P is the distance in the sense of (*) from the point $P(x, t)$ to the boundary of D lying in the half-space $t < \tau$; $d_{PQ} = \min(d_P, d_Q)$.

Theorem 4. Let $u(x, t)$ have continuous derivatives with respect to x up to order $2m$ and a continuous derivative with respect to t , and satisfy the parabolic equation

$$\sum_{|k| \leq 2m} a_k(x, t) D_x^k u(x, t) - D_{tu}(x, t) = f(x, t).$$

Suppose that $|a_k|_{-\alpha(r), \alpha(r)}$, $|k| = 2m$, and $|a_k|_{2m-|k|+B\alpha(r)-\alpha(r), \alpha(r)}$, $|k| < 2m$, are bounded by the constant $\tilde{K}$. Suppose that $|f|_{2m+B\alpha(r)-\alpha(r), \alpha(r)} < \infty$, $\alpha(r) \in \mathcal{A}_1$. Then

$$|u|_{0, 2m+B\alpha(r)} \leq K \left(|f|_{2m+B\alpha(r)-\alpha(r), \alpha(r)} + |u|_0 \right),$$

where

$$|u|_{0, 2m+B\alpha(r)} = \sum_{|k| < 2m} \sup_{P \in D} d_P^{|k|} |D_x^k u(P)| + \sum_{|k|=2m} |D_x^k u|_{2m, B\alpha(r)} + |D_{tu}|_{2m, B\alpha(r)}$$

and the constant K depends on the constant of parabolicity, $\tilde{K}$, n , the diameter of the domain D , and the function $\alpha(r)$.

For $\alpha(r) \in \mathcal{A}_2$, Schauder-type estimates have been obtained for solutions of the first boundary-value problem for parabolic systems in the sense of Petrovsky. For $\alpha(r) = \lambda$, $\lambda \in (0, 1)$, these results were obtained in papers ^(7,8).

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