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1967

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Abstract

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UDC 513.881

MATHEMATICS

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ON THE ε -ENTROPY OF ARZELÀ COM- PACTA OF FUNCTIONS DEFINED ON CLOSED SETS OF POSITIVE LEBESGUE MEASURE

(Presented by Academician A. N. Kolmogorov on 27 I 1967)

Let A be an arbitrary bounded closed set of n -dimensional Euclidean space R_n . Any nondecreasing continuous and subadditive function $\omega(t)$, defined on the half-axis $0 \leq t < \infty$, determines the class $D_\omega^A(C)$ of all real functions $f(x) = f(x_1, \dots, x_n)$, given at the points $x \in A$, which satisfy the condition

$$|f(x') - f(x'')| \leq \omega(|x' - x''|) \quad (x', x'' \in A)$$

and the condition

$$|f(x)| \leq C \quad (x \in A).$$

The equality $\lim_{t \rightarrow 0} \omega(t) = 0$, by Arzelà's theorem, is equivalent to the compactness of $D_\omega^A(C)$ in the uniform metric

$$\rho(f, \varphi) = \max_{x \in A} |f(x) - \varphi(x)|,$$

and the rate at which $\omega(t)$ decreases to zero as $t \rightarrow 0$ in general characterizes the measure of massiveness of the compactum $D_\omega^A(C)$ in this theorem.

In the well-known work of A. N. Kolmogorov ⁽¹⁾, in connection with the problem of superpositions of functions having prescribed smoothness (see ⁽²⁾), and in the spirit of the ideas of information theory, as a measure of the massiveness of a compactum Q in a metric space R , there were introduced its absolute ε -entropy $H_\varepsilon(Q)$, the ε -entropy $H_\varepsilon^R(Q)$ of the set Q relative to R , and the ε -capacity $\mathcal{E}_\varepsilon(Q)$ (see ⁽³⁾, § 1). In the same article ⁽¹⁾ the problem was posed of investigating the growth of these characteristics as $\varepsilon \rightarrow 0$ for various compacta Q occurring in analysis; the inequality

$$\mathcal{E}_{\varepsilon 2}(Q) \leq H_{\varepsilon}(Q) \leq H_{\varepsilon}^R(Q) \leq \mathcal{E}_{\varepsilon}(Q),$$

was given, its role in such an investigation was shown, and exact orders of growth of the ε -entropy (ε -capacity) were obtained for certain important cases.

One of the first results of the corresponding table of A. N. Kolmogorov (see ⁽¹⁾, estimate III, 2) consists in the following: if A is some n -dimensional cube in R_n , and $\omega(t) = t^{\alpha}$ ($0 < \alpha \leq 1$), then the following relation of order holds*:

$$\mathcal{E}_{\varepsilon}\{D_{\omega}^A(C)\} \asymp H_{\varepsilon}\{D_{\omega}^A(C)\} \asymp \varepsilon^{-n/\alpha} \quad (\omega(t) = t^{\alpha}). \quad (2)$$

Further investigations, connected with the attempt to generalize this result to more massive functional compacta $D_{\omega}^A(C)$ arising when one considers Arzelà characteristics $\omega(t)$ that decrease slowly to zero, or other compact-in-themselves sets A (see ⁽⁴⁾, § 17; ⁽³⁾, § 9; ⁽⁵⁾, theorem 8; ⁽¹¹⁾, pp. 3, 27), even for the simplest compacta $A \subset R_n$ (an interval, a rectangle, a parallelepiped), left open

* Here and below we use the notation adopted by N. Bourbaki in ⁽⁹⁾, Chap. 5, for comparing infinitely small and infinitely large quantities.

the question of the exact order of growth of the ε -entropy $H_{\varepsilon}\{D_{\omega}^A(C)\}$ and the ε -capacity $\mathfrak{E}_{\varepsilon}\{D_{\omega}^A(C)\}$ in these cases, reducing only to estimates of the form (see ⁽⁴⁾, § 17)

$$2^{H_{2\omega^{-1}(2\varepsilon)}(A)} \asymp \mathfrak{E}_{2\varepsilon}\{D_{\omega}^A(C)\} \asymp H_{\varepsilon}\{D_{\omega}^A(C)\} \asymp 2^{H_{1/2\omega^{-1}(\varepsilon/2)}(A)} \quad (3)$$

under the assumption that A is connected, and to estimates of the form (see ⁽³⁾, § 9, p. 77)

$$2^{H_{2\omega^{-1}(2\varepsilon)}(A)} \asymp \mathfrak{E}_{2\varepsilon}\{D_{\omega}^A(C)\} \asymp H_{\varepsilon}\{D_{\omega}^A(C)\} \asymp 2^{H_{1/2\omega^{-1}(\varepsilon/2)}(A)} \log \frac{1}{\varepsilon} \quad (4)$$

without this assumption.

The estimate (3), due to A. G. Vitushkin ⁽⁴⁾, makes it possible to obtain the order of growth of the ε -entropy $H_{\varepsilon}\{D_{\omega}^A(C)\}$ for connected compact sets $A \subset R_n$, generally speaking, only when $\omega^{-1}(2\varepsilon) \asymp \omega^{-1}(\varepsilon)$. Even in the case where A is a finite interval of the real axis, it is not difficult to give examples showing that without this additional restriction on the massiveness of $D_{\omega}^A(C)$ the extreme terms of inequality (3) may turn out to be infinitely large quantities of different orders.

In connection with the problem under consideration, the author ⁽⁷⁾ established the following

Theorem 1. *If a connected compact set A in a metric space R satisfies the condition*

$$2^{H_\varepsilon(A)} \asymp 2^{H_\tau(A)} \quad (5)$$

for $\varepsilon \asymp \tau$, then always

$$0 < \lim_{\varepsilon \rightarrow 0} H_\varepsilon\{D_\omega^A(C)\} \cdot 2^{-H_{\omega^{-1}(2\varepsilon)}(A)} < \infty, \quad (6)$$

$$\overline{\lim}_{\varepsilon \rightarrow 0} H_\varepsilon\{D_\omega^A(C)\} \cdot 2^{-H_{\omega^{-1}(2\varepsilon)}(A)} < \infty, \quad (7)$$

where

$$\omega(t_1) + \omega(t_2) \leq 2\omega\left(\frac{t_1 + t_2}{2}\right). \quad (8)$$

In particular, if the closed bounded set A belonging to the space R_n is connected and

$$H_\varepsilon(A) = n \log_2 1/\varepsilon + O(1) \quad (9)$$

(for example, if A is an n -dimensional parallelepiped), then, when condition (8) is satisfied, the relation of exact order is always valid (see ⁽⁶⁾)

$$\mathfrak{E}_{2\varepsilon}\{D_\omega^A(C)\} \asymp H_\varepsilon\{D_\omega^A(C)\} \asymp \left\{ \frac{1}{\omega^{-1}(2\varepsilon)} \right\}^n, \quad (10)$$

showing that the order in the right-hand estimate (3) is crude if $\omega(t)$ tends to zero sufficiently slowly.

The last result, giving exact orders of the ε -entropy and ε -capacity of arbitrarily massive Arzelà compacta of real functions $f(x_1, \dots, x_n)$ of n real variables and revealing an essential difference in their growth as $\varepsilon \rightarrow 0$, naturally leads to the question of what the sets $A \subset R_n$ are for which (10) always holds. For such sufficiently massive compacta $Q = D_\omega^A(C)$, in the orders of growth of the corresponding terms of inequality (1) as $\varepsilon \rightarrow 0$ there appear arbitrarily large gaps, whose absence in other cases plays the decisive role in obtaining the known estimates (see ⁽³⁾).

Theorem 2. *For relation (10) to hold it is necessary, and when condition (8) is satisfied it is sufficient, that the compact set $A \subset R_n$ have positive n -dimensional Lebesgue measure.*

The proof of this assertion is based on Theorem 1, the consideration of Hausdorff p -measures ⁽¹⁰⁾, and the extension of functions with preservation of their modulus of continuity and maximum modulus.

Theorem 2 shows that the requirement that the set $A \subset R_n$ be connected is not dictated by the nature of the question. In particular, the result of A. N. Kolmogorov ⁽²⁾, for any natural n , remains valid for all closed sets $A \subset R_n$ having positive n -dimensional Lebesgue measure, and only for them. For $n = 1$ this conclusion also follows from the estimate recently obtained by Vosburg ⁽⁸⁾ for one-dimensional compacta A and $\omega(t) = t^\alpha$ ($0 < \alpha \leq 1$),

$$H_\varepsilon\{D_{t^\alpha}^A(1)\} \asymp N_\delta(A) \log\{2\varepsilon^{-1}[N_\delta(A)]^{-\alpha}\} + \log \frac{1}{\varepsilon},$$

where $N_\delta(A) = 2^{H_\delta(A)}$, $\delta = \varepsilon^{1/\alpha}$, if one uses the particular case $n = 1$ (see ⁽⁸⁾) of the following general proposition.

Theorem 3. *For any natural n , the asymptotic equality (9) for a bounded closed set $A \subset R_n$ holds if and only if the n -dimensional Lebesgue measure of A is positive.*

The last proposition complements the first of the estimates given in the above-mentioned table of A. N. Kolmogorov ⁽¹⁾ (see ⁽³⁾, Theorem VIII). In the case when the Lebesgue measure of the boundary of the set $A \subset R_n$ is zero (the set A has “volume”), excluding, for example, everywhere disconnected sets of positive measure, the asymptotic estimate of $H_\varepsilon(A)$ and $C_\varepsilon(A)$ as $\varepsilon \rightarrow 0$ is known in a somewhat more precise form than (9) (see ⁽³⁾, § 4, Theorem IX).

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Received
27 I 1967

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