

# LIMIT TRANSITION FOR NONSINGULAR SYSTEMS

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**Abstract**

**Full Text**

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*MATHEMATICS*

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## LIMIT TRANSITION FOR NONSINGULAR SYSTEMS

$$\dot{X} = A_n(t)X$$

(Presented by Academician I. N. Vekua on 9 XII 1966)

1. On a finite interval  $[a, b]$  we consider linear systems of ordinary differential equations in matrix form

$$\dot{X} = A(t)X, \quad X(a) = I \quad (a \leq t \leq b), \quad (1)$$

$$\dot{X}_n = [A(t) + R_n(t)]X_n, \quad X_n(a) = I, \quad n = 1, 2, \dots \quad (a \leq t \leq b). \quad (2)$$

As usual, it is assumed that  $X(t)$ ,  $X_n(t)$  are absolutely continuous and that relations (1), (2) hold for almost all  $t$  from  $[a, b]$ . The matrices  $A(t)$ ,  $R_n(t)$  are assumed summable on  $[a, b]$ :  $\|A(t)\|_L < \infty$ ,  $\|R_n(t)\|_L < \infty$  (finite dimensionality is immaterial; see below). We shall use the following notation:

$$\|C(t)\|_L = \int_a^b \|C(s)\| ds, \quad C^\vee(t) = \int_a^t C(s) ds;$$

$C_n(t) \Rightarrow C(t)$  means that  $\max_{a \leq t \leq b} \|C_n(t) - C(t)\| \rightarrow 0$  ( $n \rightarrow \infty$ ).

In the present paper we study the conditions under which

$$X_n(t) \Rightarrow X(t) \quad (n \rightarrow \infty). \quad (3)$$

Obviously, (3) is equivalent to the following: for any fixed initial conditions, the solutions (matrix or vector) of the systems  $\dot{U}_n = (A + R_n)U_n$  converge uniformly on  $[a, b]$  to the corresponding solution of the system  $\dot{U} = AU$ . Therefore the fulfillment of (3) means, so to speak, the "convergence" of the system  $\dot{U} = (A + R_n)U$  to the system  $\dot{U} = AU$ . The well-known condition sufficient for (3),

$\|R_n(t)\|_L \rightarrow 0$ , proves in many questions to be too rigid. It is natural to suppose that more delicate conditions for (3) should, in their character, be close to the condition

$$R_n^\vee(t) \Rightarrow 0 \quad (n \rightarrow \infty), \quad (4)$$

and this is indeed so. As simple examples show, (4) by itself is neither necessary nor sufficient for (3); however, under certain additional assumptions, (3)  $\Leftrightarrow$  (4) (see Theorem 3).

2. The following reduction theorem is very essential.

**Theorem 1.** *Relation (3) holds if and only if  $Y_n(t) \Rightarrow I$  ( $n \rightarrow \infty$ ), where*

$$\dot{Y}_n = R_n(t)Y_n, \quad Y_n(a) = I, \quad n = 1, 2, \dots \quad (a \leq t \leq b). \quad (5)$$

In order not to carry out the proof in both directions, while retaining (5), let us consider, alongside (1), (2), another sequence of systems of the same type

$$\dot{U} = [A(t) + B(t)]U, \quad \dot{U}_n = [A(t) + B(t) + R_n(t)]U_n,$$

$$U(a) = U_n(a) = I, \quad n = 1, 2, \dots,$$

with an arbitrary summable on  $[a, b]$   $B(t)$ , and show that from (3) there follows the relation  $U_n(t) \Rightarrow U(t)$ . This will prove the independence of (3) from  $A(t)$ , which is asserted in the theorem.

**Lemma 1.** If  $\|\dot{V}(t)\| \leq \|C(t)\| \|V(t)\|$ ,  $V(a) = I$ , then

$$\|V(t) - I\| \leq \exp \left( \int_a^t \|C(s)\| ds \right) - 1 \quad (t \geq a).$$

This simple observation follows from the inequalities

$$\begin{aligned} \|V(t)\| \leq w(t) &= 1 + \|V(t) - I\| \leq 1 + \int_a^t \|C(s)\| \|V(s)\| ds \leq \\ &\leq 1 + \int_a^t \|C(s)\| w(s) ds \quad (t \geq a), \end{aligned}$$

if one uses the Gronwall-Bellman estimate (see <sup>(6)</sup>) for  $w(t)$ .

Suppose that relation (3) is satisfied, or, what is the same thing,

$$J_n(t) = X_n(t)X^{-1}(t) \Rightarrow I \quad (n \rightarrow \infty) \quad (6)$$

(3)  $\Leftrightarrow$  (6), since  $\|X\|, \|X^{-1}\| \leq \exp \|A\|_L$ . Define the matrices  $Z_n(t)$  by the formula

$$U_n(t) = J_n(t)U(t)Z_n(t) \quad (J_n = X_nX^{-1}), \quad n = 1, 2, \dots \quad (7)$$

It is enough to show that  $Z_n \Rightarrow I$ , whence the relation  $U_n \Rightarrow U$  will follow by virtue of (6), (7) and the estimate  $\|U\| \leq \exp \|A + B\|_L < \infty$ . We find:

$$\dot{J}_n = (A + R_n)X_nX^{-1} - X_nX^{-1}A = (A + R_n)J_n - J_nA,$$

$$\dot{U}_n = (A + B + R_n)J_nUZ_n = \dot{J}_nUZ_n + J_n\dot{U}Z_n + J_nU\dot{Z}_n =$$

$$= (A + R_n)J_nUZ_n - J_nAUZ_n + J_n(A + B)UZ_n + J_nU\dot{Z}_n,$$

$$(BJ_n - J_{nB})UZ_n = J_nU\dot{Z}_n, \quad \dot{Z}_n = U^{-1}J_n^{-1}(BJ_n - J_{nB})UZ_n,$$

$$\|Z_n(t) - I\| \leq \exp \left( \int_a^t \|U^{-1}\| \|J_n^{-1}\| \|BJ_n - J_{nB}\| \|U\| ds \right) - 1 \leq$$

$$\leq \exp \left( 2 \int_a^t \|U^{-1}(s)\| \|U(s)\| \|J_n^{-1}(s)\| \|B(s)\| \|J_n(s) - I\| ds \right) - 1 \quad (a \leq t \leq b). \quad (8)$$

We have used Lemma 1 and the inequality

$$\|BJ_n - J_{nB}\| = \|B(J_n - I) - (J_n - I)B\| \leq 2\|B\| \|J_n - I\|.$$

Since  $\|U\|, \|U^{-1}\| \leq \exp \|A + B\|_L < \infty$ ,  $\|B\|_L < \infty$ ,  $\|J_n(t)\| \Rightarrow 1$ ,  $\|J_n(t) - I\| \Rightarrow 0$ , by virtue of (6), the exponential integral, and along with it the entire right-hand side of (8), tend to zero uniformly on  $[a, b]$  as  $n \rightarrow \infty$ . Thus,  $Z_n \Rightarrow I$ . The theorem is proved.

3. Thus, the question of the validity of (3) is reduced to the study of systems (5), which are usually considerably simpler to investigate than (2), and in many cases admit direct integration. An interesting example of this kind is provided by systems corresponding to scalar equations of order  $m$ . Suppose that on  $[a, b]$  there are given the equations

$$x^{(m)} + p_1(t)x^{(m-1)} + \dots + p_m(t)x = 0, \quad (9)$$

$$x_n^{(m)} + [p_1(t) + r_{1n}(t)]x_n^{(m-1)} + \dots + [p_m(t) + r_{mn}(t)]x_n = 0, \quad n = 1, 2, \dots, \quad (10)$$

with coefficients summable on  $[a, b]$ . In the usual way (see (3)), rewriting (9), (10) in matrix form (1), (2), we obtain matrices  $R_n(t)$  of order  $m$  with all rows zero except the last, which has the form  $(-r_{mn}, \dots, -r_{1n})$ . For such  $R_n$  (as for any triangular matrices) the systems (5) are obviously integrated by quadratures. It is easy to see that the first  $m-1$  rows of the matrices  $Y_n(t)$  coincide with the corresponding rows of the identity matrix, while the last row has the form

$$\begin{aligned} & - \int_a^t r_{mn}(u) \exp \left( \int_t^u r_{1n}(s) ds \right) du, \dots \\ & \dots, - \int_a^t r_{2n}(u) \exp \left( \int_t^u r_{1n}(s) ds \right) du, \exp \left( - \int_a^t r_{1n}(s) ds \right). \end{aligned}$$

The last element tends uniformly on  $[a, b]$  to 1 if and only if  $r_{1n}^\vee(t) \Rightarrow 0$ . Thus, for equations (9), (10) the reduction theorem leads to the following result of a definitive character.

**Theorem 2.** *In order that, for any fixed initial conditions, the solutions of equations (10) converge uniformly on  $[a, b]$ , together with their derivatives up to order  $m-1$ , to the corresponding solution of equation (9), it is necessary and sufficient that*

$$\int_a^t r_{1n}(s) ds \Rightarrow 0, \quad \int_a^t r_{in}(u) \exp \left( \int_a^u r_{1n}(s) ds \right) du \Rightarrow 0, \quad i = 2, \dots, m \quad (n \rightarrow \infty).$$

Let us note that the fulfillment of the conditions  $r_{in}^\vee(t) \Rightarrow 0$ ,  $i = 1, 2, \dots, m$ , does not ensure convergence not only in  $C^{m-1}[a, b]$ , but even in  $C[a, b]$ . The equations

$$\ddot{x}_n + p_n(t)\dot{x}_n + q_n(t)x_n = 0, \quad \text{where } p_n(t) = (n^{\alpha-1} + n^{-1} \sin nt)^{-1} \cos nt,$$

$$q_n(t) = -n^\beta \sin nt, \quad 0 < \alpha < \beta < 1$$

may serve as an illustration. Although  $p_n^\vee(t) \Rightarrow 0$ ,  $q_n^\vee(t) \Rightarrow 0$ , one can show that for any nonzero initial data  $|x_n(t)| \rightarrow \infty$  on  $(a, b]$  as  $n \rightarrow \infty$ . If, however, the

perturbation of the coefficient at  $x^{(m-1)}(t)$  is absent, i.e.  $r_{1n}(t) \equiv 0$  ( $n = 1, 2, \dots$ ), then the condition

$$r_{in}^\vee(t) \Rightarrow 0, \quad i = 2, \dots, m \quad (n \rightarrow \infty) \quad (11)$$

is necessary and sufficient for convergence in  $C^{m-1}[a, b]$  also for the matrix equations of order  $m$  (9), (10). This also follows from Theorem 1, since in this case the systems (5) are again integrated directly.

**4. Theorem 3.** Suppose that, as  $n \rightarrow \infty$ , the matrices  $R_n(t)$  satisfy at least one of the conditions

$$1) \|R_n\|_L^\vee < c; \quad 2) \|R_n R_n^\vee\|_L \rightarrow 0; \quad 3) \|R_n^\vee R_n\|_L \rightarrow 0;$$

$$4) \|R_n R_n^\vee - R_n^\vee R_n\|_L \rightarrow 0.$$

Then relation (4) is necessary and sufficient for (3).

In the part concerning sufficiency, condition 1) is more restrictive than the others; otherwise 1)-4) are pairwise independent. Each of the conditions 2)-4) is fulfilled, in particular, for systems corresponding to equations (possibly matrix ones) (9), (10), if  $r_{1n}(t) \equiv 0$  (in this case  $R_n R_n^\vee = R_n^\vee R_n \equiv 0$ ); thereby we again arrive at (11).

We omit the proof. It is based on Theorem 1, and in the sharpest case 4) also on the following estimate.

**Lemma 2.** At every point of differentiability of the matrix  $F(t)$ ,

$$\left\| \frac{d}{dt} e^{F(t)} - \dot{F}(t) e^{F(t)} \right\| \leq \frac{1}{2} e^{\|F(t)\|} \|\dot{F}(t) F(t) - F(t) \dot{F}(t)\|.$$

Indeed, putting  $K = \dot{F}F - F\dot{F}$ ,  $C_{mn} = F^{m-1} \dot{F} F^{n-m}$  ( $1 \leq m \leq n$ ), we have

$$\|C_{r+1,n} - C_{rn}\| = \|F^{r-1} K F^{n-r-1}\| \leq \|K\| \|F\|^{n-2},$$

$$\|C_{mn} - C_{1n}\| \leq \|C_{mn} - C_{m-1,n}\| + \dots + \|C_{2n} - C_{1n}\| \leq$$

$$\leq (m-1) \|K\| \|F\|^{n-2},$$

$$\left\| \frac{d}{dt} e^F - \dot{F} e^F \right\| = \left\| \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{m=2}^n (C_{mn} - C_{1n}) \right\| \leq$$

$$\leq \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{m=2}^n (m-1) \|K\| \|F\|^{n-2} = \frac{1}{2} \|K\| e^{\|F\|}.$$

5. In conclusion we shall touch on possible generalizations and modifications. The extension of what has been presented to equations of the form  $\dot{X} = XA_n(t)$  or  $\dot{X} = A_n(t)X + F_n(t)$ ,  $\|F_n - F\|_L \rightarrow 0$ , is quite obvious. The finiteness of  $[a, b]$  was not used in Theorems 1, 3, so that they remain valid also for  $b = \infty$  (however, the condition  $\|A\|_L, \|R_n\|_L < \infty$  then becomes very restrictive). Theorem 2 does not hold for  $b = \infty$ , as is shown by the example  $\ddot{x}_n + n^{-1}t^{-2}x_n = 0$  ( $1 \leq t < \infty$ ). Everything presented, except for Theorem 2, extends to equations in a Banach space with Bochner-integrable operators  $A(t)$ ,  $R_n(t)$  (see <sup>(2,7)</sup>): the role of the matrix equation is then played, as is known (see <sup>(5)</sup>), by the corresponding equation in the ring of bounded operators. In general, one may regard (1), (2) as equations in some normed ring with identity.

Some of the estimates given above find application in other questions. Thus, estimate (8), in combination with Hölder's inequality, is useful in the study of other kinds of convergence of solutions besides uniform convergence. Lemma 2 makes it possible to prove the following assertion: *if the matrix  $A(t)$  is locally summable on  $[a, b)$ ,  $b \leq \infty$ , the integral*

$$\int_t^b A(s) ds = B(t)$$

*converges, and  $\|AB - BA\|_L < \infty$ , then the matrix equation  $\dot{X} = A(t)X$  has a solution of the form  $X(t) = I + o(1)$  ( $t \rightarrow b$ ). This supplements Wintner's results on linear asymptotic equilibrium <sup>(1)</sup> (see also <sup>(4)</sup>), which contain the conditions  $\|AB\|_L < \infty$  or  $\|BA\|_L < \infty$ .*

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*Note: Figure translations are in progress. See original paper for figures.*

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