

ON THE EXISTENCE OF A COUNTABLE SET OF PERIODIC MOTIONS IN A NEIGHBORHOOD OF A HOMOCLINIC CURVE

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Abstract

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MATHEMATICS

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ON THE EXISTENCE OF A COUNTABLE SET OF PERIODIC MOTIONS IN A NEIGHBORHOOD OF A HOMOCLINIC CURVE

(Presented by Academician L. S. Pontryagin, 11 III 1966)

§ 1. Consider a system of $m + n + 1$ differential equations

$$\frac{dz}{dt} = Z(z), \quad (1)$$

where the right-hand sides are at least twice continuously differentiable in some domain $G \subset R^{n+m+1}$. Suppose that the system has a periodic motion $\mathfrak{L}_0$ of period 2π , whose characteristic exponents satisfy the following properties: except for one zero exponent, there are $m \geq 1$ exponents $\lambda_1, \dots, \lambda_m$ having negative real parts, and $n \geq 1$ exponents $\gamma_1, \dots, \gamma_n$ having positive real parts. As is known, in this case there exist two integral manifolds of dimensions $m + 1$ and $n + 1$, which intersect along $\mathfrak{L}_0$. We shall denote the stable integral manifold by $\mathfrak{M}^+$, and the unstable one by $\mathfrak{M}^-$.

Suppose that there exists a trajectory Γ_0 , doubly asymptotic to $\mathfrak{L}_0$. Following Poincaré ⁽¹⁾, who indicated the possibility of the existence of such motions (see also ⁽⁶⁾), we shall call Γ_0 a homoclinic curve. Since $\Gamma_0 \subset \mathfrak{M}^+ \cap \mathfrak{M}^-$, it is natural to assume with respect to this intersection that it is rough, i.e.,

$$\dim(W_M^+ \cap W_M^-) = 1, \quad (2)$$

where W_M^+ and W_M^- denote the tangent spaces to $\mathfrak{M}^+$ and $\mathfrak{M}^-$ at a point $M \in \Gamma_0$.

Theorem 1. *If condition (2) is fulfilled, then in every neighborhood of the homoclinic curve Γ_0 there is contained a countable number of periodic motions of saddle type.*

Remark 1. It follows from this theorem that the five well-known roughness conditions for dynamical systems, formulated by Smale ⁽³⁾ as hypotheses, are not independent.

Remark 2. As is known, a nonautonomous system

$$dz/dt = f(z, t)$$

with periodic right-hand side is reduced to the study of the autonomous system

$$\begin{aligned} dz/dt &= f(z, \theta), \\ d\theta/dt &= 1 \end{aligned}$$

with toroidal phase space, which has a global section.

Birkhoff ⁽²⁾ proved for the case of a point mapping of the plane into the plane preserving area that in a neighborhood of a homoclinic point there is contained a countable set of periodic points. However, Birkhoff's method cannot be extended to the general case.

Our proof of Theorems 1 and 2 does not use the construction of a global section.

Remark 3. Other cases of the existence of a countable set of periodic motions connected with a nontransversal intersection of integral manifolds are considered in the works ^(10–12). A number of Smale's works ^(4,5) are devoted to questions connected with the roughness of homeomorphisms having a countable set of periodic motions.

The investigation of the roughness and structure of a homoclinic cell is the subject of a work of Yu. I. Neimark and the author (see ⁽⁷⁾).

§ 2. The proof of Theorem 1 is based on the construction of a point mapping in a neighborhood of Γ_0 , using the so-called parametric representation of the mapping in a neighborhood of $\mathfrak{L}_0$. (On the parametric representation of a mapping see also ⁽¹²⁾.)

With the help of a certain change of variables, system (1) in a neighborhood of $\mathfrak{L}_0$ can be brought to the form

$$\begin{aligned} dx/dt &= Ax + P(x, y, \theta)x, \\ dy/dt &= By + Q(x, y, \theta)y, \\ d\theta/dt &= 1, \end{aligned} \tag{3}$$

where A and B are constant matrices of orders m and n , reduced to normal Jordan form, whose characteristic roots are respectively $\lambda_1, \dots, \lambda_m$ and $\gamma_1, \dots, \gamma_n$; P and Q are periodic functions of θ , in the general case of period 4π , which vanish for $x = y = 0$. The equations of $\mathfrak{M}_+$ and $\mathfrak{M}_-$ in these variables have respectively the form

$$y = 0, \quad x = 0. \tag{4}$$

Denote by S_0 the surface

$$\|x\|^2 = r^2, \quad \|y\|^2 < r^2,$$

and by S_1

$$\|y\|^2 = r^2, \quad \|x\|^2 < r^2.$$

Lemma 1. *For all sufficiently small $r > 0$ and $0 \leq \theta \leq 4\pi$, the surfaces S_0 and S_1 are surfaces without contact for the trajectories of system (3).*

It follows from Lemma 1 that a trajectory l , passing through the point $M_0(x_0, y_0, \theta_0) \in S_0$, where $\|y_0\| \neq 0$, after a time t_0 will intersect S_1 at some point $M_1(x_1, y_1, \theta_1)$. Denote this mapping by T_0 .

Let

$$\begin{aligned} x(t) &= x(t, x_0, y_0, \theta_0), \\ y(t) &= y(t, x_0, y_0, \theta_0), \\ \theta(t) &= t + \theta_0 \end{aligned}$$

be the equations of l . Then T_0 can be written in the form

$$\begin{aligned} x_1 &= x(t_0, x_0, y_0, \theta_0), \\ y_1 &= y(t_0, x_0, y_0, \theta_0), \\ \theta_1 &= t_0 + \theta_0 \pmod{\tau} \quad (\tau \text{ is equal either to } 2\pi \text{ or to } 4\pi), \end{aligned} \tag{5}$$

where the transition time t_0 is determined from the equation

$$\|y(t_0, x_0, y_0, \theta_0)\|^2 = r^2.$$

It is not difficult to prove that the mapping T_0 can be written in the form

$$\begin{aligned} x_1 &= x^p(t_0, x_0, y_1, \theta_0), \\ y_1 &= y^p(t_0, x_0, y_1, \theta_0), \\ \theta_1 &= t_0 + \theta_0 \pmod{\tau}, \quad \|x_0\|^2 = r^2, \quad \|y_1\|^2 = r^2, \end{aligned} \tag{6}$$

where x^p and y^p tend to zero as $t_0 \rightarrow \infty$, together with all first derivatives. In the parametric form (6), the mapping T_0 is defined for all x_0, y_1 satisfying the condition $\|x_0\|^2 = r^2$, $\|y_1\|^2 = r^2$, and $t_0 > 0$.

Let $M_0^+(x_0^+, 0, \theta_0^+)$ and $M_1^-(0, y_1^-, \theta_1^-)$ be the points of intersection of Γ_0 with S_0, S_1 , and let U_0, U_1 be their neighborhoods on S_0, S_1 :

$$U_0 = [M_0(x_0, y_0, \theta_0) : \|x_0 - x_0^+\| < \varepsilon, \|y_0\| < \varepsilon, |\theta_0 - \theta_0^+| < \varepsilon],$$

$$U_1 = [M_1(x_1, y_1, \theta_1) : \|x_1\| < \delta, \|y_1 - y_1^-\| < \delta, |\theta_1 - \theta_1^-| < \delta].$$

Let ε and δ be such that the vectors x_0 and y_1 can be represented in the form $x_0 = (\tilde{x}_0, x'_0)$, $y_1 = (\tilde{y}_1, y'_1)$, where $\tilde{x}_0$ is expressed through the components of the $(m-1)$ -dimensional vector x'_0 , and $\tilde{y}_1$ through the components of the $(n-1)$ -dimensional vector y'_1 .

From the theorem on continuous dependence on initial conditions it follows that, along trajectories close to Γ_0 , one can establish a one-to-one correspondence between certain neighborhoods of the points M_0^+ and M_1^- . Denote this mapping by T_0 . Let δ be such that $T_1 U_1 \subset U_0$. The mapping T_1 in the variables under consideration can be written in the form

$$\begin{aligned}\bar{x}'_0 &= F(x_1, y'_1, \theta_1), \\ \bar{y}_0 &= G(x_1, y'_1, \theta_1), \\ \bar{\theta}_0 &= \Phi(x_1, y'_1, \theta_1),\end{aligned}$$

or

$$\begin{aligned}\Delta \bar{x}'_0 &= A_1 x_1 + B_1 \Delta y'_1 + C_1 \Delta \theta_1 + \dots, \\ \bar{y}_0 &= A_2 x_1 + B_2 \Delta y'_1 + C_2 \Delta \theta_1 + \dots, \\ \Delta \bar{\theta}_0 &= a x_1 + b \Delta y'_1 + c \Delta \theta_1 + \dots,\end{aligned}\tag{8}$$

where $\Delta \bar{x}'_0 = \bar{x}'_0 - (x'_0)^+$, $\Delta y'_1 = y'_1 - (y_1^-)'$, $\Delta \bar{\theta}_0 = \bar{\theta}_0 - \theta_0^+$, $\Delta \theta_1 = \theta_1 - \theta_1^-$.

Fulfillment of condition (2) means that

$$\text{rang } \|B_2 C_2\| = n.\tag{9}$$

From Lemma 1, (9), and the parametric representation (6) of the mapping T_0 , it follows that $\bar{y}_0 \neq 0$ can be represented in the form

$$\bar{y}_0 = y^\Pi(\bar{t}_0, \bar{x}_0, \bar{y}_1, \bar{\theta}_0),\tag{10}$$

where $\bar{t}_0$ is the time of passage of the phase point from the point $\bar{M}_0(\bar{x}_0, \bar{y}_0, \bar{\theta}_0) \in S_0$ to the point $\bar{M}_1(\bar{x}_1, \bar{y}_1, \bar{\theta}_1) \in S_1$.

Consider the mapping $T = T_1 T_0$. It can be written in the form

$$\begin{aligned}\bar{x}'_0 &= F(x^\Pi(t_0, x_0, y_1, \theta_0), Y'_1, t_0 + \bar{\theta}_0), \\ y^\Pi(\bar{t}_0, \bar{x}_0, \bar{y}_1, \bar{\theta}_0) &= G(x^\Pi(t_0, x_0, y_1, \theta_0), Y'_1, t_0 + \bar{\theta}_0), \\ \bar{\theta}_0 &= \Phi(x^\Pi(t_0, x_0, y_1, \theta_0), y'_1, t_0 + \bar{\theta}_0).\end{aligned}\tag{11}$$

It is easy to see that it is defined for $\|\Delta x'_0\| < \varepsilon$, $\|\Delta y'_1\| < \delta$, $|\theta_0 - \theta_0^+| < \varepsilon$, $|t_0 + \theta_0 - \theta_1^-| < \delta \pmod{4\pi}$, and maps the point $(x'_0, y'_1, t_0, \theta_0)$ to the point $(\bar{x}'_0, \bar{y}'_1, \bar{t}_0, \bar{\theta}_0)$. Representing t_0 in the form $t_0 = \tau k - \theta_0^+ + \theta_1^- + \tau_0$, we obtain

that the domain of definition of T consists of a countable union of disjoint domains

$$\sigma_k = [\|\Delta x'_0\| < \varepsilon, \|\Delta y'_1\| < \delta, |\theta_0 - \theta_0^+| < \varepsilon, |\tau_0 + \Delta\theta_0| < \delta],$$

where $k = \bar{k}, \bar{k} + 1, \dots$

Thus the problem of the existence of a countable set of periodic motions in $U(\Gamma_0)$ has been reduced to the problem of finding fixed points of the mapping T and of its powers.

Lemma 2. *The mapping T has a countable number of fixed points of saddle type.*

The fixed points of the mapping T are found from the system

$$\begin{aligned} f &\equiv x'_0 - F(x^\Pi, y'_1, \theta_0 + \tau_0 + \theta_0^+) = 0, \\ g &\equiv y^\Pi - G(x^\Pi, y'_1, \theta_0 + \tau_0 + \theta_0^+) = 0, \\ \varphi &\equiv \theta_0 - \Phi(x^\Pi, y'_1, \theta_0 + \tau_0 + \theta_0^+) = 0. \end{aligned} \quad (12)$$

Using the condition

$$\lim \frac{D(f, g, \varphi)}{D(x'_0, y'_1, \theta_0, \tau_0)} = (-1)^{n-1} |B_2 C_2| \neq 0, \quad (13)$$

we easily prove that the mapping T in each σ_k , for $k \geq \bar{k}' \geq \bar{k}$, has a fixed point M_k^* .

Linearizing (11) in a neighborhood of an equilibrium point and forming the characteristic equation, we are convinced that each equilibrium point is a rough point of saddle type.

Let N_p be the set of equilibrium points of the mapping T of multiplicity p , i.e. equilibrium points of the mapping T^p . In an analogous manner it is shown that N_p is nonempty for all $p \geq 2$.

From the construction of the mapping T it follows that in any neighborhood of the homoclinic curve Γ_0 there is contained a countable set of periodic motions.

§ 3. Suppose that system (1) has periodic motions $\mathfrak{L}_1, \mathfrak{L}_2, \dots, \mathfrak{L}_k$ of saddle type and trajectories $\Gamma_1, \Gamma_2, \dots, \Gamma_l$, where $l \geq k$, such that $\alpha(\Gamma_i) = \omega(\Gamma_{i+1})$, $\alpha(\Gamma_0) = \omega(\Gamma_l)$, where by $\alpha(\Gamma_i)$ and $\omega(\Gamma_i)$ are denoted the α - and ω -limit sets of the trajectory Γ_i , which are the indicated periodic solutions. Let $U(\Gamma_1, \dots, \Gamma_l)$ be a certain neighborhood of the contour $\Gamma_1, \dots, \Gamma_l$, which we shall call a homoclinic contour.

Analogously to Theorem 1, one proves

Theorem 2. *If the stable and unstable manifolds of the periodic motions intersect along Γ_i roughly, then in any neighborhood of the homoclinic contour*

$U(\Gamma_1, \dots, \Gamma_l)$ there is contained a countable set of periodic motions of saddle type.

Here the proof reduces to the investigation of the fixed points of the mapping T , representable in the form of a product of mappings $T_1^{(i)}T_0^{(i)}$, where $T_0^{(i)}$ is the mapping in a neighborhood of $\mathfrak{L}_i$, which is constructed analogously to the mapping T_0 of § 2, and $T_1^{(i)}$ is the mapping in a neighborhood of Γ_i , which is constructed analogously to the mapping T_1 .

Remark. The study of the mapping T makes it possible to establish the existence of an invariant set analogous to that described by Smale (13).

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Note: Figure translations are in progress. See original paper for figures.

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